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COMMUNICATION CIRCUIT FUNDAMENTALS

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BY CARL E. SMITH

APPLIED MATHEMATICS

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COMMUNICATION NETWORKS

AUDIO AND RADIO FACILITIES

COMMUNICATION CIRCUIT FUNDAMENTALS

for Radio and Communication Engineers

by CARL E. SMITH, B.S., M.S., E.E.

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COMMUNICATION CIRCUIT FUNDAMENTALS

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To

HANNAH

PREFACE

COMMUNICATION CIRCUIT FUNDAMENTALS is the second of four books designed for a complete course in radio and communication engineering, the first of which, APPLIED MATHEMATICS, contains prerequisite material necessary to complete understanding of the text presented here. More advanced treatment is planned for the last two books of the group, covering the subjects of Communication Networks and Audio and Radio Facilities.

The material of this and other books of the group is a portion of the advanced radio and communication engineering course prepared by the author for home study. The texts are intended to serve as study fundamentals in residence or correspondence courses in which a plan of *directed study* helps the student to learn the subject matter correctly and rapidly. However, for those who do not have time to take advantage of available courses, these books should prove valuable for reading and reference purposes.

This first edition of COMMUNICATION CIRCUIT FUNDAMENTALS has been preceded by several private editions used in a practical home-study course. Its preparation was started in 1934 in an effort to meet the needs of radio operators, broadcast technicians, and chief engineers who wished to increase their knowledge and equip themselves for further responsibility and advancement. The course in which this material was presented has been taken by many broadcasting station chief engineers and by many college graduates interested in specializing in the field of communication engineering. The results of this course of study have been very gratifying and serve to indicate the usefulness of the text application in technical schools and colleges that offer courses in radio and communication engineering.

The course of study covered in the four books of this group may be pursued by students with a high-school education or its equivalent. While the first book, APPLIED MATHEMATICS, correlates the previous training of the student with the specific calculations necessary in such subjects as the requirements of simultaneous equations in mesh circuits and quadratic equations in complex circuits, this text covers the physics of circuit elements and presents the fundamentals of d-c and a-c circuits. The first fifteen chapters are a prerequisite for the advanced treatment in the third volume on communication networks, while the

entire book (particularly the last five chapters on vacuum tubes) is a prerequisite for the material to be presented in the fourth book on audio and radio facilities.

The general approach is to discuss the circuit constants and then to use them in a treatment of d-c circuits. The fundamental principles of a-c circuits are introduced by a study of magnetism, inductance, and capacitance. Ohm's and Kirchhoff's laws are applied first in solving relatively simple d-c problems while later on in the text the same types of circuits are studied with the application of vector notation of a-c circuits.

The author is indebted to Kenneth Sherman for the preliminary work done on Chap. 1 and also to Charles Merchant for assisting in the revision and writing of Chaps. 5, 6, 7, 15, and 20. Mr. Merchant has read the manuscript and has made many helpful suggestions. The author also wishes to thank Calvin Warner and Edith Drake for assistance in preparing the final manuscript. Finally, the author wishes to express appreciation to his friends and the many students of the Cleveland Institute for their assistance and encouragement in the preparation and the correction of errors in COMMUNICATION CIRCUIT FUNDAMENTALS.

The illustrations in this book were made possible by the splendid cooperation of the following companies: General Electric Company, National Carbon Company, Willard Storage Battery Company, Edison Electric Company, Leeds and Northrup Company, and the Weston Electrical Instrument Corporation.

CARL E. SMITH

CLEVELAND, OHIO
November, 1949

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CHAPTER 1

MATTER AND ENERGY

In order to understand electrical systems it is helpful to have some knowledge of the composition of matter, especially from the viewpoint of the electron theory. This theory can be thought of as a mental picture of electricity. The electron theory can be used to analyze electric and magnetic circuits and determine in advance how the various components will work when they are connected together.

1.1. Nature of Matter and Energy. The physical world is made up of matter and energy. Under certain conditions these two parts can be transformed from one to the other. *Matter can be defined as anything that occupies space and has mass.* All matter is found in one of three states: solid, liquid, or gas.

Matter is not a continuous structure but actually consists of innumerable particles. By definition, *the atom of an element is the smallest particle of matter that still retains all the chemical and physical properties of the element* as it is divided and subdivided. At present, the chemical theory of matter is known as the *atomic theory.* According to this theory, all matter is composed of elemental atoms of which there are 96 different types. Each of these 96 elements has different chemical and physical properties.

Energy can be defined as that which causes a change in matter. It is the ability to do work. Energy is not of material existence, that is, it has no mass and does not occupy space. For illustration, the light from the sun, the heat from a stove, and the electricity in an electric power line are forms of energy.

1.2. The Atom. According to the atomic theory, atoms are small particles of matter made up of various arrangements of protons and electrons. The difference between atoms of different elements must be in the arrangement and number of these particles. The *electron* is the natural elementary quantity of negative electricity. It is the smallest unit of matter known to science and is found in every element. The *proton* is the natural elementary quantity of positive electricity. It is thought to be a positron plus a neutron. The *positron* has practically the same mass as an electron, but the charge is positive. The *neutron* has practically the same mass as the proton but is uncharged.

One of the most popular concepts of the atom is due to Bohr. He

likened the atom to our solar system, the sun being the nucleus and the planets the electrons. The simplest atom, that of the gas hydrogen, which is the first in the chemist's periodic table, is thought to be made up of one positive charge or proton, neutralized by one negative charge or electron, leaving the atom electrically neutral (see Fig. 1.1). A neutral atom can be defined as an atom made up of equal amounts of positive and negative electricity. Furthermore, the mass of the proton is 1,846 times that of the electron; hence the normal hydrogen atom is 1,847 times heavier than the electron.

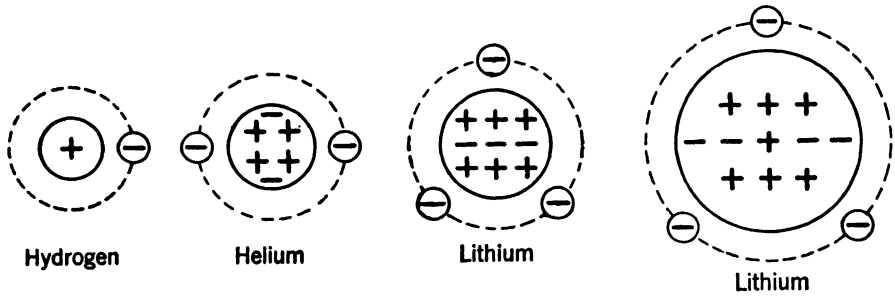


FIG. 1.1. Electronic construction of some simple atoms.

An atom of a given element whose nucleus is of different mass from that of the normal atom is called an *isotope*. About 1 out of every 5,000 hydrogen atoms has a nucleus containing a neutron along with a proton. The combination is known as a *deuteron*. This makes *heavy-hydrogen* atoms 3,694 times heavier than the electron. The physical properties of isotopes differ very slightly from those of the normal atoms.

The amount of positive charge on the nucleus of an atom determines the number of orbital electrons as well as the chemical and physical properties of the element. The number of orbital electrons in a normal atom is indicated by the *atomic number* of the element. The atomic numbers of the elements vary from 1 to 96.

The element with the atomic number 2 is the gas helium. This atom is thought to be built up of a center nucleus of two protons and two neutrons with two more electrons revolving around the nucleus. The two electrons moving around the nucleus are generally called *orbital* electrons, since they revolve in orbits whose center is the nucleus of the atom. It can be seen that the nucleus may be made up of both protons and neutrons, there being sufficient orbital electrons in each neutral atom to offset the nuclear protons. In the case of helium, there are two protons, two neutrons,

and two orbital electrons. The general agreement between the atomic and electron theories is shown by the extract from the periodic table shown in Table 1.1. It will be noted that in general the atomic weight is nearly twice the atomic number. Also, the atomic number corresponds to the number of orbital electrons. The variations in the atomic weights from twice the atomic number are believed to be caused by the presence of two or more possible atomic structures, or isotopes, all of which differ slightly in nuclear structure but have the same number of orbital electrons that are required for a balanced atom. Figure 1.1 shows two possible combinations, each with three orbital electrons, one being heavier than the

TABLE 1.1

Name	Atomic weight	Atomic number	Orbital electrons
Hydrogen	1.008	1	1
Helium	4.003	2	2
Lithium	6.940	3	3
Beryllium	9.020	4	4
Boron	10.820	5	5
Carbon	12.011	6	6
Nitrogen	14.008	7	7
Oxygen	16.000	8	8

other owing to a more complex nuclear structure. The atomic weight of lithium is 6.94 because the lighter lithium atom, having a weight of 6.00, occurs much less frequently in nature than the heavier lithium atoms.

In order to obtain a better picture of the protons and electrons in an atom, imagine a circle $\frac{1}{8}$ in. in diameter spaced 100 miles from another circle 21 ft in diameter. These circles would represent the comparative size of the elements in a hydrogen atom, the larger circle being the electron, the smaller the proton, and the 100 miles the relative distance between them as the electron revolves around the nucleus. Physically, the radius of the electron has been found by Millikan to be 1.9×10^{-13} cm, the mass 8.98×10^{-28} g, and the charge 1.59×10^{-19} coulomb. Or from another angle consider that in 1 g of hydrogen, there are 6.02×10^{23} atoms of hydrogen.

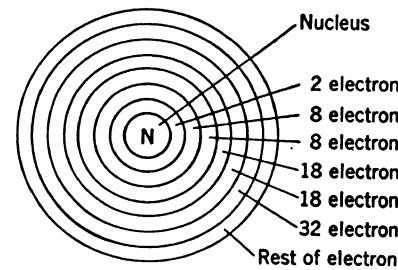


FIG. 1.2. Electronic shells or orbits in an atom.

For convenience, the nucleus can be considered as a sphere with electrons moving about within concentric shells that are somewhat greater in diameter than the nucleus. Consider Fig. 1.2 as a cross section of concentric spherical shells. It will be noted that there are 2 electrons in the

first shell, 8 electrons in the second and third shells, 18 in each the fourth and fifth shells, 32 in the sixth; for elements having more than 86 electrons, the remaining electrons go into the seventh shell. In the case of hydrogen there is only 1 electron in the first shell instead of 2; if an atom has 27 electrons, there will be 9 electrons in the fourth shell instead of 18.

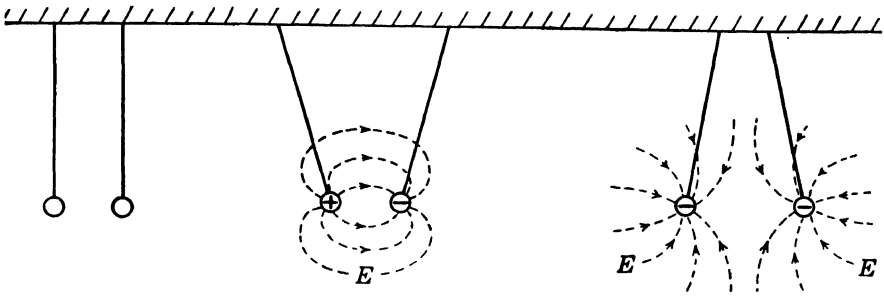
1.3. Conductors and Insulators. With the concept that matter is composed of positive and negative particles or charges of electricity, the electrical properties of materials can now be considered.

In some elements, such as copper, silver, and platinum, some of the electrons are rather loosely attached to the positive nucleus in the atom; many electrons are free to move about from atom to atom, or between the atoms within the body of the metal. In a neutral state, for each "free" electron moving around in the body, there is an atom lacking one electron. This atom that is lacking an electron is positively charged. In some cases, these free electrons move around between the atoms, which of themselves are constantly moving; or they may strike an atom, lodging themselves in that atom's structure, and displacing another electron, which continues the electron movement or drift.

This condition of free electrons moving about within the physical limits of a body is found to some extent in all elements or compounds. Those metals such as silver, copper, and aluminum are known as "good conductors." They have many free electrons in a given volume of the metal. Conversely, those elements and compounds which have but few free electrons are known as "poor conductors," "nonconductors," or "insulators." In other words, the only difference between a good conductor and an insulator is in the number of free electrons in a given volume. If an insulator is heated, so as to increase the velocity of its electrons, some of them are often liberated from their orbits. This increases the number of free electrons, and may make the insulator become a partial conductor. Thus, an element or compound may be a conductor under certain conditions, and an insulator under other conditions.

1.4. Electric Charges and Lines of Force. Any quantity of electricity is known as a *charge*. If a conductor or insulator has an excess of electrons, it is said to have a negative charge. Similarly, if it has a deficiency of electrons, the protons will be in excess and it will have a positive charge. This is shown in Fig. 1.3. If two uncharged bodies are suspended by strings as shown in Fig. 1.3*a*, there will be no deflection. If the bodies have unlike charges, they will be attracted as shown in Fig. 1.3*b*, while if they have like charges they will be repelled, as shown in Fig. 1.3*c*. In Fig. 1.3*c* the charged bodies each have an excess of electrons, hence are repelled. They would also be repelled if they each had a deficiency of electrons so that the bodies were both positive.

The attraction or repulsion can be explained by studying the electric lines of force. There are electric lines of force emanating from the proton or positive charge, and if no other charges are present, these lines of force extend out to infinity in all directions. The lines of force about an electron or negative charge in free space are similar, except that the lines of force come *in* from infinity in all directions and terminate on the electron. When a positive charge is brought near a negative charge, a number of the lines of force from the positive charge terminate on the negative charge



(a) No charge no effect (b) Unlike charges attract (c) Like charges repel

FIG. 1.3. Lines of force used to represent fields of force.

as shown in Fig. 1.3*b*. These lines of force act like rubber bands tending to pull the charges closer together. Now, if similar charges are brought close together as shown in Fig. 1.3*c*, the electric lines of force repel each other and thus react on the charges to push them farther apart. These lines of force associated with electrons and protons are used to explain many electrical phenomena.

1.5. Methods of Liberating Electrons from Atoms. Electrons can be liberated from atoms in several ways as follows:

a. Heat. This method is used in modern vacuum tubes. The filament or cathode is heated enough to give the electrons sufficient velocity to overcome the forces holding them in their atomic orbits.

b. Friction. Another very common method of removing electrons is by friction. When a rod of hard rubber is rubbed with a piece of fur, electrons are removed from the fur and deposited upon the rubber. This leaves a positive charge on the fur, which means that there are not so many negative charges as there are proton positive charges. The hard rubber, on the other hand, has an excess of electrons, which means that the proton positive charges are all neutralized with electrons and there are still some electrons left over. This terminology is used throughout electrical work, a positive point being a point with an insufficient number of electrons and a negative point having an excessive number of electrons.

Insulating bodies such as rubber and fur will retain their charges for some time because electrons find it difficult to move about on an insulator.

c. Spontaneous Disintegration by Radioactivity. Some elements are so unstable that they are continually giving off electrons without any external energy being applied. An example of this is the element radium, which is continually emitting not only electrons, or *beta rays*, as they are called, but also *alpha rays*, which consist of the helium nucleus, and *gamma rays*, which are X rays. This *radioactivity* or giving off of nuclear particles eventually changes the atomic structure of the element. Radium is believed to change through many forms, the last or end product being the element lead.

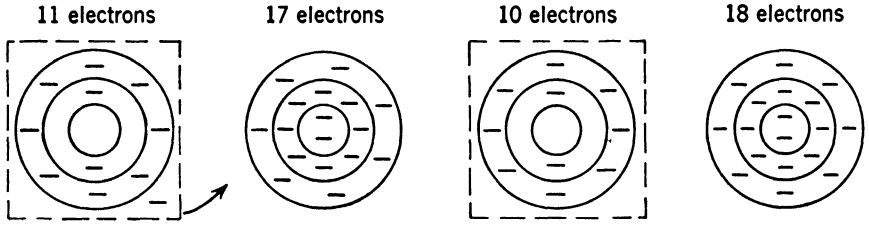


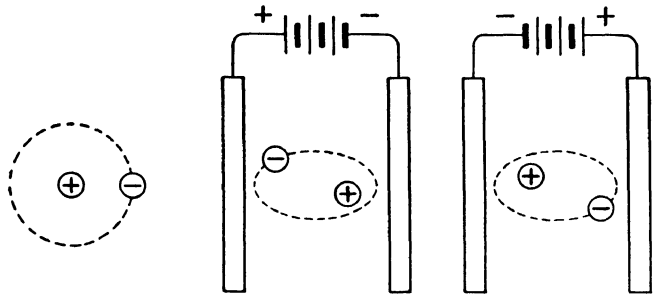
FIG. 1.4. Common salt atoms.

d. Ionization. Electrons are also removed from atoms in some cases when elements and compounds are dissolved in water or other solvents. A very common example of this is the *ionization* of salt when it is dissolved in water. Chemists have proved that the molecules are broken up into positively and negatively charged particles called *ions*. This ionization takes place when any acid, base, or salt is dissolved in the proper solvent. In the case of common salt (sodium chloride), the electronic construction of the atoms is believed to be as shown in Fig. 1.4. One molecule of salt is believed to have 28 orbital electrons, of which the highly active sodium atoms contains 11, the remaining being in the also active chlorine atom. Sodium activity is thought to be due to the fact that the sodium atom can readily give up an electron, while chlorine is active due to its ability to attract an extra electron. When the salt molecule is dissolved in water, the sodium loses one electron to the chlorine atom, which is then charged negatively and is known as a *stable negative ion.* The sodium atom, being short one electron, has a positive charge and is known as a *stable positive ion.*

The principle of ionization is used to make a battery. If two different plates are properly chosen and inserted in a suitable solution, there results a difference in potential between the two plates. Without going into detail at this time about batteries, this electrode-to-solution potential is

illustrated by the common storage battery, consisting of a lead plate and a lead oxide plate in a solution of sulfuric acid. When the battery is fully charged, the potential of the positive plate with respect to the solution is +2.5, that of the negative plate is −0.1, resulting in a difference of potential of 2.6 volts. When the battery is discharged, the positive-plate potential is +2.05 and the negative-plate potential is +0.25, resulting in a potential difference of 1.8 volts.

e. Voltage. Electrons can be removed from atoms in some cases by applying a voltage or difference of potential. It is generally assumed that the path of an electron in an atom is circular under normal conditions, as shown in Fig. 1.5*a*. This electron path or orbit can be distorted



(a) Simple Atom with undistorted orbit (b) Orbit distorted due to charge on plates (c) Orbit distortion with reversed charge

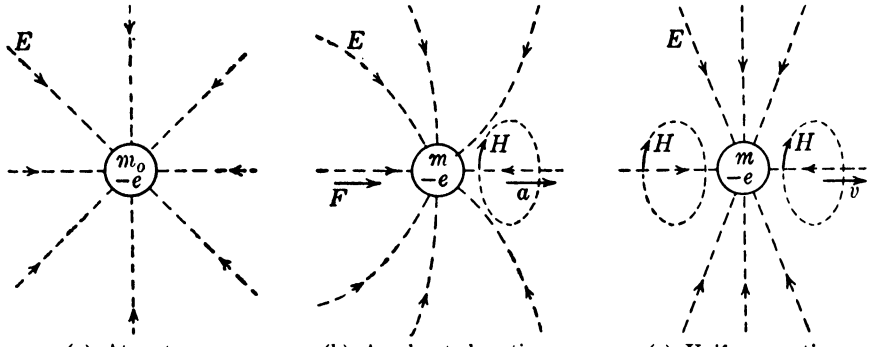
FIG. 1.5. Electron orbit distortion in an electric field.

by placing the atom in an electric field, as shown in Figs. 1.5*b* and *c*. A common example of the strain in an atom due to a potential difference is found in the insulation of a capacitor.

When the voltage between two capacitor plates is increased, or the thickness of the insulation is decreased, the stress on the atoms increases; that is, the forces on the orbital electrons trying to force them out of their orbits are increased. If the potential is increased sufficiently, the electron is attracted out of the orbit, leaves the atom, and moves to the positive plate. At the same time, other electrons are stressed to the point where they can leave the orbits in their atoms, and the movement of one electron releases enough energy to free electrons in other atoms, resulting in the disrupting of the insulator, or in popular terminology, the insulator "flashes through," or "breaks down," and becomes a conductor.

1.6. Electric and Magnetic Fields of an Electron. When an electron is stationary, the electric lines of force extend radially in all directions as shown in Fig. 1.6*a*. Now if this electron is accelerated, the electric lines of force are distorted as shown in Fig. 1.6*b*. The electric lines of force must always terminate on the electron, hence they follow the electron in

a definite manner. However, this distortion of the electric field creates another field known as the *magnetic field*, with magnetic lines of force forming circles around the electron as shown in Fig. 1.6*b*. The direction of electron motion is along the axis of the circle of the magnetic lines of force. As long as the electron accelerates, the number of magnetic lines of force will increase. When the electron velocity is constant, the magnetic lines of force are constant, and the electric lines of force are distributed uniformly as shown in Fig. 1.6*c*.



(a) At rest (b) Accelerated motion (c) Uniform motion

FIG. 1.6. Electrostatic and electromagnetic fields of an electron.

According to relativity theory, the stationary electron can be visualized as a spherical shell as shown in Fig. 1.6*a*. This shell can be considered to contract in the direction of motion, as shown in Figs. 1.6*b* and *c*, by a definite ratio depending upon the velocity. According to Einstein's relativity theory, the mass of the electron varies with the velocity according to the equation

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1.1}$$

where m = mass of the electron, g

$m_0 = 9 \times 10^{-28}$ g = rest mass of the electron

v = electron velocity, cm/sec

$c = 3 \times 10^{10}$ cm/sec = velocity of light

From this equation we see that as the velocity increases, the radical term becomes smaller and will go to zero if $v = c$. This means that the mass of the electron increases as the velocity increases and would go to infinity if the velocity of the electron were equal to the velocity of light.

An illustration of fast-moving electrons is found in cathode rays, where the electrons have sufficient energy to cause X rays that will penetrate through such material things as steel, wood, flesh, and bone with enough

energy left to make a chemical effect on a film. Much use is made of this in the practical field for inspecting opaque materials.

Example 1.1. Determine the mass of an electron in a beta ray traveling at 90 per cent of the velocity of light. What is the percentage increase in its mass as compared to its mass when at rest?

Solution. Substituting in Eq. (1.1)

$$m = \frac{9 \times 10^{-28}}{\sqrt{1 - \left(\frac{2.7 \times 10^{10}}{3 \times 10^{10}}\right)^2}} = 2.066 \times 10^{-27} \text{ g} \qquad \textit{Ans.}$$

$$\frac{m - m_0}{m_0} 100 = \frac{2.064 \times 10^{-27} - 9 \times 10^{-28}}{9 \times 10^{-28}} 100 = 128.5\% \text{ increase in mass} \qquad \textit{Ans.}$$

1.7. Energy of the Electron. Lorentz was first to advance the proposal that an electron might be electromagnetic in nature. He reasoned thus: When an electron is caused to be accelerated, the magnetic field created by its motion is increased; any magnetic field represents energy; expenditure of energy is needed to expand the field because of the charge alone. Hence an electrical charge must have some sort of mass, which must be related to the energy expended. This approach has been generalized by the relativity theory to show that mass and energy are related by

$$U = mc^2 \tag{1.2}$$

where U = energy, ergs
m = mass of the electron, g
$c = 3 \times 10^{10}$ cm/sec = velocity of light

Example 1.2. Determine the energy of an electron due to its mass alone (*a*) at rest and (*b*) if its velocity is 90 per cent of the velocity of light.

Solution. (*a*) At rest

$$U_0 = m_0c^2 = (9 \times 10^{-28})(9 \times 10^{20}) = 8.1 \times 10^{-7} \text{ erg} \qquad \textit{Ans}$$

(*b*) At 90 per cent of the velocity of light, from Example 1.1,

$$U = mc^2 = (2.066 \times 10^{-27})(9 \times 10^{20}) = 1.854 \times 10^{-6} \text{ erg} \qquad \textit{Ans.}$$

From the above line of reasoning it is evident that energy and matter are closely related. Perhaps they are one and the same thing. If so, everything is electricity. The various elements found on the earth are just different ways of putting this energy into packages having distinct chemical and physical properties.

Summary

All matter is composed of electrons and protons, infinitesimal-sized particles that are conceived to be part mass, part energy. The atom consists of a central nucleus containing protons and neutrons (except for hydrogen, which has only one proton) and

an outer ring or group of orbital electrons. Ninety-six different types of atomic structures exist, each having different chemical and physical properties.

Matter can exist in three different states: solid, gaseous, or liquid, depending upon conditions of pressure and temperature.

Neutral atoms have positive protons and negative electrons in equal amounts. Atoms that can lose one or more outer electrons easily form substances known as "conductors." Atoms from which electrons are not easily detached are found in materials called "insulators."

Electrons may be removed from atoms by heating, friction, chemical action, or a high potential force.

Conductors and insulators differ in the number of "free" electrons they contain.

A body that has an excess of electrons over the number required for each "neutral" atom is said to be charged *negatively.* If some of the electrons have been removed from the neutral atoms and there are not enough electrons in the body to make all atoms neutral, then it is said to be charged *positively.*

An ion is a charged unit that exists in acids, bases, and salts when dissolved in certain solvents.

If a potential is applied across an insulator, the electrons of the atoms in the insulator are "strained" or distorted in their orbits. Energy is absorbed when the orbit is strained and released when it returns to normal.

When excessive potential is applied to an insulator, the strain is so great that electrons are released and the insulator passes current and "breaks down."

Exercises

1.1. (*a*) What properties do all elements have in common? (*b*) What properties do all atoms have in common?

1.2. How are the orbital electrons in an atom related to the atomic weight of the element?

1.3. What is the difference between a conductor and an insulator?

1.4. What happens when a substance that is capable of forming ions is immersed in a dissolving solution?

1.5. What is a stable ion? Explain.

1.6. What happens to an orbital electron of an atom when the atom is placed between two oppositely charged plates?

1.7. Draw a diagram showing the relative positions of the electrons in a copper atom, which contains 29 electrons.

1.8. What determines the chemical and physical properties of an element?

1.9. What is the mass of an electron having a velocity of 5×10^7 cm/sec?

1.10. What is the energy of an electron having a velocity of 5×10^7 cm/sec?

CHAPTER 2

ELECTRICAL QUANTITIES

In the first chapter the discussion was quite general in regard to the nature of matter and energy. Many important topics were discussed briefly to give a qualitative background on which a quantitative treatment can now be developed.

This chapter will deal with some of the fundamental electrical quantities and discuss the elements of an electric circuit, giving particular attention at this time to d-c circuits.

2.1. Electric Charge. The Coulomb. The practical system of electrical units can be based upon the attraction or repulsion of electric charges. Coulomb determined experimentally that the force between two charges varied (1) directly as the product of the two charges, (2) inversely as the square of the distance between the two charges, and (3) inversely as the dielectric constant K of the medium between the charges. In equation form Coulomb's law can be written

$$F = \frac{Q_1 Q_2}{K d^2} \tag{2.1}$$

where F = force acting between the two charged bodies, dynes
Q_1 = units of charge on first body
Q_2 = units of charge on second body
d = distance of separation, cm
K = dielectric constant ($K = 1$ for air)

This equation defines the unit of charge in terms of force, distance, and the dielectric constant. Thus, *a unit charge is that charge which, if placed one centimeter from a similar charge in free space, is repelled with a force of one dyne.* This unit of charge is called the *statcoulomb*, the unit of charge in the electrostatic or the esu system of units. This definition is explained in Fig. 2.1. The electric charge may be thought of as a physical quantity that projects itself into space and thus can by its electric field manifest itself throughout space.

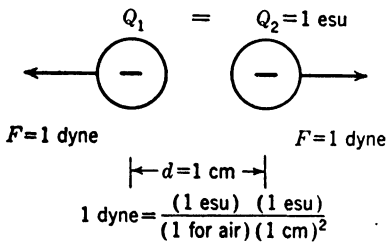


FIG. 2.1. Charges defining a statcoulomb.

The negative charges in Fig. 2.1 can be expressed in terms of electrons. If the free electrons were counted in these charges, it would be found that there are 2.0947×10^9 electrons in each charge. That is,

$$1 \text{ statcoulomb} = 2.0947 \times 10^9 \text{ electrons} \tag{2.2}$$

Example 2.1. Charges of $+16$ and -32 esu, respectively, are 16 cm apart in air. What is the force acting on each?

Solution. Substituting in Eq. (2.1)

$$F = \frac{(+16)(-32)}{(1)(16^2)} = -2 \text{ dynes} \quad \text{(a force of attraction on each charge)} \quad \textit{Ans.}$$

Example 2.2. Charges of $+200$ and $+150$ esu, respectively, are 20 cm apart in a medium whose dielectric constant is 2.5. What force acts on each charge?

Solution. Substituting in Eq. (2.1)

$$F = \frac{(+200)(+150)}{2.5(20^2)} = 30 \text{ dynes} \quad \text{(a force repelling each charge)} \quad \textit{Ans.}$$

The statcoulomb is an extremely small unit of charge, and for most practical work the unit used is the *coulomb*, *one coulomb being equal to* 3×10^9 *statcoulombs*, or

$$1 \text{ coulomb} = 6.289 \times 10^{18} \text{ electrons}$$

2.2. Electric Current. The Ampere. *Electric current* can be defined as the drift of electrons through a conductor. Or to put it differently, whenever there is a general flow of electrons past a given point, it is said that there is an electron current at that point. In the practical system of electrical units, this electron flow can be expressed quantitatively in terms of *amperes*. *An ampere of current is defined as the flow of one coulomb of charge per second of time.* In terms of electrons, when 1 amp of current is flowing in a conductor, there is a drift of 6.289×10^{18} electrons past the given point each second. In equation form

$$\begin{aligned} 1 \text{ amp} &= 1 \text{ coulomb/sec} \\ &= 6.289 \times 10^{18} \text{ electrons/sec} \end{aligned} \tag{2.3}$$

This large number of electrons moving past a point in 1 sec can be expressed in terms of the average velocity of the electrons in the conductor and the number of electrons per unit volume; thus, mathematically,

$$i = nve \tag{2.4}$$

where i = current density, amp/sq cm
n = number of electrons/cu cm
v = average drift velocity of the electrons, cm/sec
$e = 1.59 \times 10^{-19}$ coulomb/electron

If n is considered as the number of electrons in a unit length of conductor, the area does not have to be mentioned, providing all these electrons drift past the measuring point at the velocity v.

It is of interest to determine the amount of actual matter carried in an electron current. Consider for illustration a generator delivering 4,000 amp through the conductors connected to a load. What is the weight of the electrons passing through such conductors in 1 day? The effective mass of an electron is

$$m = 8.98 \times 10^{-28} \text{ g}$$

Then the weight of the electrons in 1 amp flowing for 1 sec is

(Mass of an electron)(number of electrons in 1 amp/sec)

$$= (8.98 \times 10^{-28})(6.289 \times 10^{18})$$

$$= 56.49 \times 10^{-10} = 5.649 \times 10^{-9} \text{ g/sec for 1 amp}$$

Since there are 86,400 sec in 1 day, we have

Total weight

= (number of sec)(number of amp/sec)(weight of 1 amp/sec)

Substituting

$$\text{Total weight} = (86{,}400)(4{,}000)(5.649 \times 10^{-9})$$

$$= 1.95 \text{ g of electrons per day} \qquad \textit{Ans.}$$

This gives an idea as to the weight of the electron itself.

It must be remembered that when an atom loses an electron, as in the case of a conductor with many free electrons, that atom still retains its chemical and physical properties, with the additional property of being charged positively. When electron current is flowing, the atoms themselves do not drift along the wire. The current is due to the electron drift, while the atoms themselves form the substance of the conductor.

When a current is passed through an ionized solution, the situation is different, there being no free electrons, but only charged atoms or molecules called "ions." In this case, the atoms or ions themselves travel through the solution, giving up their charges to the electrode and depositing out on that electrode or uniting chemically with other substances in the solution. Since the ions are just charged atoms, their mass is relatively high, and their velocity extremely low, in contrast to the velocity of the free electrons in a conductor. One obvious proof of the fact that the atoms as stable ions actually move through the solution carrying the electrons that constitute the electric current is found in electroplating, where copper is removed from one electrode and deposited on another. This sort of thing does not happen when current is passed through a metal conductor.

The rate at which an electric charge passes a point can be detected and

measured with great precision in terms of chemical, magnetic, and thermal effects. The ampere is defined in terms of its chemical effects because results of this kind can be duplicated in any part of the world within an accuracy of 1 part in 10,000. *The international ampere is defined as that steady current which when passed through a silver nitrate solution will deposit silver at the rate of* 0.001118 *gram per second.* The magnetic effect of the current is used in most d-c meters and in some a-c meters. Thermal meters are used quite widely to measure high-frequency alternating currents.

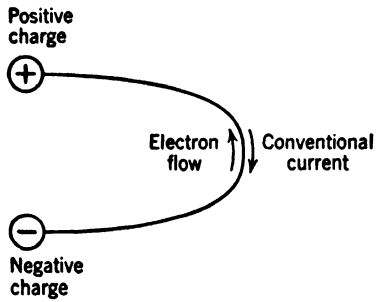


FIG. 2.2. Fictitious "conventional" current and actual "electron" flow illustrated.

Conventional current was defined long before the electron theory, and unfortunately the early experimenters assumed the current to be a drift of positive charges and hence a flow in the opposite direction to electron flow or electron current as defined here.

To summarize: (1) *Conventional current* flows from positive to negative in an external circuit. (2) *Electron current* flows from negative to positive in an external circuit. These facts are diagramed in Fig. 2.2.

Example 2.3. A No. 12 gauge copper wire carries 20 amp. What is the average velocity of electron drift if 1 cu cm contains 8.47×10^{23} free electrons?

Solution. From Table 2.3 on page 28, the current density is

$$i = \frac{\text{amperes}}{\text{area in sq cm}} = \frac{\text{amperes}}{\text{sq in.} \times 2.54^2}$$

$$= \frac{20}{0.005129 \times 6.4516} = 604 \text{ amp/sq cm}$$

By Eq. (2.4)

$$v = \frac{i}{ne} = \frac{604}{(8.47 \times 10^{23}) \times (1.59 \times 10^{-19})}$$

$$= 0.00448 \text{ cm/sec} = \text{average velocity} \qquad \textit{Ans.}$$

2.3. Classification of Currents. Electric currents can be broadly classified as direct currents and as alternating currents. *A direct current (d-c) can be defined as a current in which the general drift or flow of electrons is in one direction.* Figures 2.3*a*, *b*, and *c* show the various types of direct current. It will be noted in these illustrations that the current is always positive, sometimes zero, but never negative. A battery current is a good example of a continuous direct current. If the direct current varies regularly, it is called a *pulsating direct current.* Pulsating direct currents are caused by regular variations in the circuit conditions; examples are the result of a full-wave rectifier and the ripple output from the filter

following the rectifier. Transient direct currents can be produced by opening or closing circuits and in general are not repeated regularly.

An alternating current (a-c) can be defined as a current in which the electrons flow first in one direction and then in the other direction. Figures 2.3*d*, *e*, and *f* show the various types of alternating current. In these

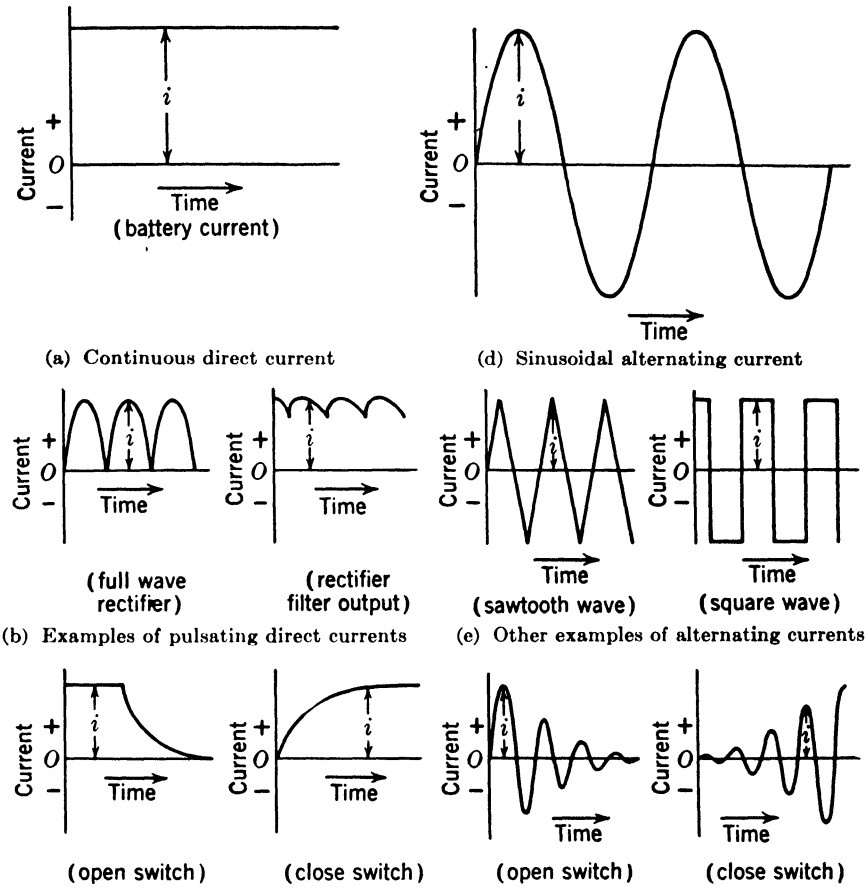


FIG. 2.3. Classification of currents.

illustrations, the current is alternately positive and negative. The simplest and most common of alternating wave forms is the sine wave. There are countless other forms of regularly occurring or steady-state alternating currents such as that shown in Fig. 2.3*e*. Finally there are the transient alternating currents resulting from opening or closing a circuit or from some other change in the circuit condition.

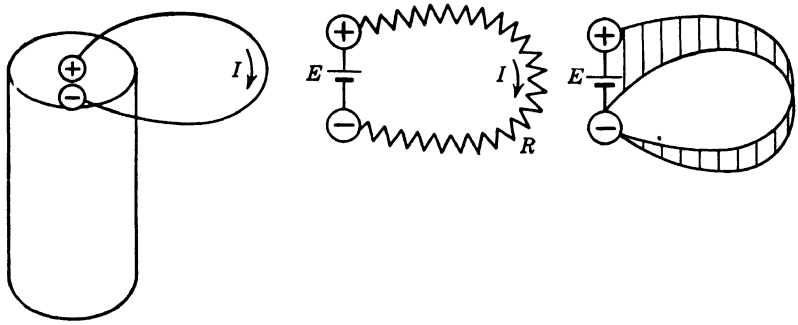
2.4. Electric Pressure. The Volt. In Fig. 2.2 a negative charge was connected to a positive charge by means of a conductor. The deficiency of electrons on the positive charge attracts electrons, and the excess of electrons on the negative charge repels electrons, with the result that electrons flow along the wire until the charges are equalized. This means that on a conductor the electrons distribute themselves uniformly so that the forces of attraction and repulsion are equalized. In other words, as long as the charges are not equalized, the electrons in the conductor are under an electric pressure to move in such a direction that the charges will be equalized. This electric pressure is commonly referred to as *potential difference* and can be defined as follows: A *potential difference is an electrical condition or difference in electric pressure that causes (or tends to cause) electrons to flow.* The potential difference between two points is measured in *volts.* When an electron current flows between two points, energy is gained or lost depending upon whether the charge is moved to a point of higher potential or to a point of lower potential. *A volt can be defined as the potential difference between two points when one joule*[1] *of energy is gained or lost in moving one coulomb of charge between the two points.*

2.5. Electromotive Force. Emf. It has already been shown that a current will flow in a conductor between charges of unequal potential. If as in Fig. 2.2 these charges have only one connection, that is, the wire connecting them, the current will flow only for a moment equalizing the charges and making the potential difference zero. In order to maintain the current in the wire, it is necessary to have a constant potential difference between the charges. So that this may happen, there must be some force that tends to separate the positive and negative electric charges. In other words there must be some motive power to maintain the charges and cause (or tend to cause) the current to flow. Such motive power can be the chemical or physical changing of energy into electrical form. A good example of a generator is a dry cell which by chemical action causes a constant emf of 1.5 volts. As another example, mechanical energy can be converted into electrical energy by a d-c generator to produce a constant emf. *The electromotive force (emf) of a generator can be defined as the work done (in volts) by the generator on each coulomb of electricity that passes through it.*

In Fig. 2.4, the battery lifts a charge (coulomb) of electricity from zero potential at the negative terminal to a positive potential of 1.5 volts at the positive terminal by means of the chemical action in the dry cell. In doing this, the battery does 1.5 joules of work on every coulomb of

[1] One joule = 10^7 ergs = 10^7 dyne-centimeters.

electrical charge passing through it. As the coulomb flows from the positive terminal of the battery to the negative terminal through the external conductor, it expends 1 joule of energy in heating the conductor. Thus, the potential along the conductor gradually drops from 1.5 volts at the positive terminal to 0 volt at the negative terminal.



(a) Dry cell as source of direct-current emf connected to a conductor

(b) Graphical representation of a battery and a conductor having resistance

(c) Vertical distance represents potential across battery and along the resistance

FIG. 2.4. Chemical source of emf and a conductor to carry the charges.

2.6. Chemical Sources of Emf. Batteries. Electric batteries are commonly used as a source of steady emf. In batteries chemical energy is transformed into electrical energy. In the *primary battery* the original materials are used up, while in the *secondary* or *storage battery* the materials can be used over and over again by recharging. The word "battery" means a combination of electrochemical cells but is often loosely used to refer to a single cell.

a. Wet Cells. When a conductor is immersed in an electrolyte such as a dilute solution of sulfuric acid, the chemical reaction at the surface of the conductor creates a potential difference between the conductor and the electrolyte. Different conductors create different potentials with respect to the electrolyte, some being positive and others negative when compared to hydrogen, as shown in Table 2.1.

For illustration, if a zinc and a copper plate are inserted in a dilute solution of sulfuric acid, the potential difference between the positive copper terminal and the negative zinc terminal will be approximately

$$+0.34 - (-0.76) = 1.1 \text{ volts}$$

(see Fig. 2.5*a*).

Wet cells of this type have three disadvantages that limit their use. First, the cell polarizes in use with a resulting high internal resistance due to a layer of hydrogen gas collecting on the copper electrode. Second, the

zinc dissolves owing to local action and is soon wasted away. Third, the electrolyte evaporates and is easily spilled.

Example 2.4. If a wet cell is made by using platinum and aluminum plates in a dilute solution of sulfuric acid, what will be the emf?

Solution. From Table 2.1, the emf will be $0.87 - (-1.34) = 2.21$ volts. *Ans.*

TABLE 2.1. ELECTROMOTIVE SERIES

	Metal	Symbol	Potential, volts
Less active ↑	Platinum	Pt	+0.87
	Silver	Ag	+0.80
	Copper	Cu	+0.34
	Hydrogen	H	0.00
	Lead	Pb	−0.13
	Nickel	Ni	−0.22
	Iron	Fe	−0.43
	Zinc	Zn	−0.76
↓ More active	Aluminum	Al	−1.34

b. Dry Cells. The common commercial form of the primary battery is made up of so-called *dry cells*, which are used in portable equipment and elsewhere where small amounts of power are needed over short periods of time. The term "dry cell" is a misnomer because moisture is essential to the cell's operation; however, the cell is sealed so that the moisture will not evaporate and the cells can be handled in a dry condition.

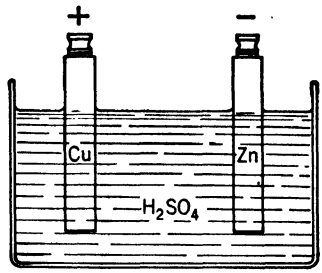
The cross-section view of a dry cell in Fig. 2.5*b* shows a carbon rod electrode in the center of a zinc cup, which is the negative electrode. The electrolyte is usually ammonium chloride mixed with some porous inert material to form a paste just inside the zinc container. Polarization is minimized by a mixture of manganese dioxide and carbon granules next to the positive electrode.

The larger dry cells are approximately 2½ in. in diameter by 6 in. tall. When the cell is in good condition, its terminal voltage is 1.5 to 1.6 volts, and it will deliver on short circuit about 30 amp. For higher voltage and very low currents, a number of smaller cells may be connected in series and mounted in a common container. Such a battery is the well-known B battery used to supply plate voltage to vacuum tubes.

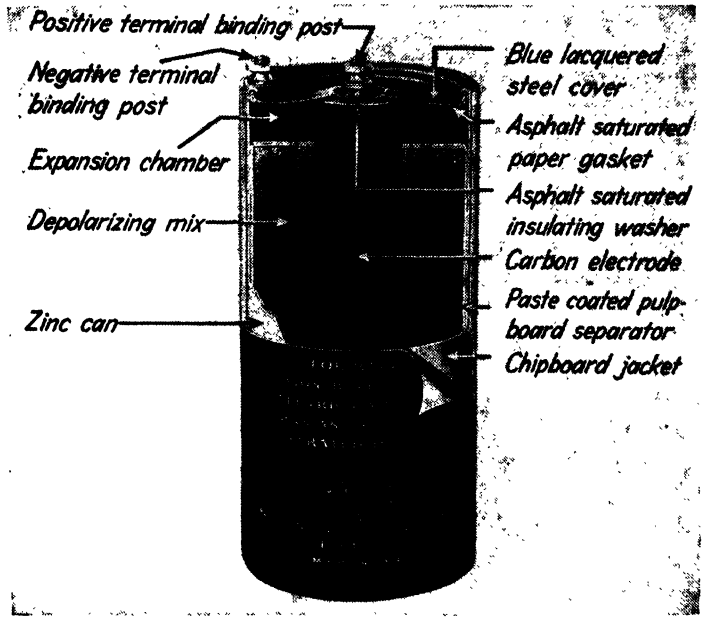
c. Standard Cell. As a legal standard of emf, the normal type of cadmium standard cell, which is a primary cell, has been adopted by international agreement. Its emf is 1.01830 volts at a temperature of 20°C. The Weston cadmium standard cell is the voltage standard of emf most widely used to calibrate electrical meters to a high degree of

accuracy. The current drain should never exceed 0.0001 amp in order to retain the voltage accuracy.

d. Lead Storage Batteries. A secondary cell that is widely used is the lead-acid cell. Commercial cells of this type consist of several positive



(a) Wet cell



(b) Dry cell

FIG. 2.5. Primary cells.

plates of brown lead peroxide separated from several negative plates of gray sponge lead, by rubber, glass-fiber, or wooden separators, and the whole assembly is immersed in a dilute solution of sulfuric acid, H_2SO_4. The portable type has a pasted plate construction consisting of a lead-and-antimony-alloy grid filled with a lead peroxide paste for the positive plate and with a pure spongy lead paste for the negative plate.

Upon discharging, both sets of plates are partially changed to lead

sulfate, $PbSO_4$, and the electrolyte to water, H_2O. Upon recharging, the cell returns to its original state. The chemical equation can be written as follows:

$$PbO_2 + Pb + 2H_2SO_4 \underset{\text{Charge}}{\overset{\text{Discharge}}{\rightleftharpoons}} 2PbSO_4 + 2H_2O$$

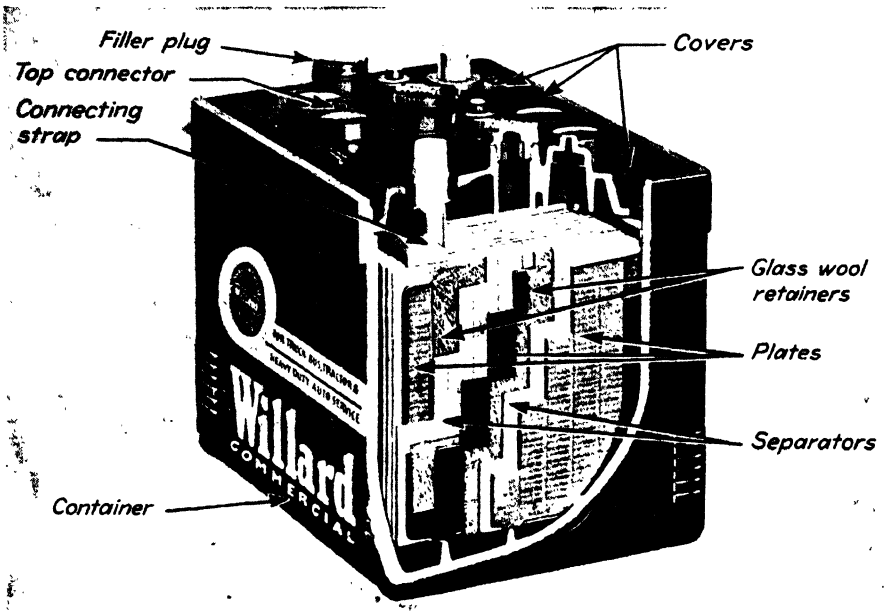


FIG. 2.6. Lead storage cell showing construction details.

For long and efficient service of the lead storage cell, the following points should be observed:

Impure electrolyte is one of the greatest sources of trouble. If impure water or acid is added, the plates may be consumed by local action with the impurities, or the plates may be dissolved by foreign acids.

Discharged condition should be avoided since in this state the lead sulfate has a tendency to harden, crystallize, and expand, with the result that the plates will buckle and the battery life will be materially reduced.

Presence of either strong or weak electrolyte has disadvantages. A strong electrolyte tends to act on the active materials and convert them to lead sulfate more rapidly when the cell is in an inactive condition. If the electrolyte solution is too weak, the emf will be reduced and the internal resistance will be increased. For the correct amount of electrolyte, the specific gravity of the charged battery should be from 1.200 to 1.300, and on discharge it should never fall below 1.170.

Overcharging and discharging should be avoided. Overcharging causes the electrolyte to boil, thus liberating hydrogen and oxygen, wasting power, and depleting the water in the battery. The rate of charging should not be great enough to cause excessive heating or violent chemical reaction. Discharging causes excessive lead sulfate, weakens the acid, and lowers the terminal voltage.

The condition of a lead storage cell can be determined by (1) specific gravity test, the value depending upon the value when fully charged, and (2) voltage when the cell is discharged at a normal rate.

A hydrometer is used for the first test, and a special low-resistance voltmeter can be used for the second type of test. When the cell is fully charged, the emf per cell is 2.1 volts and on discharge should never go below 1.75 volts.

Lead storage batteries are widely used in automobiles because of their low internal resistance and high emf. The current drain can be quite high, of the order of 200 amp, which is enough to run the starting motor in an automobile.

e. Nickel-Iron Storage Batteries. The nickel-iron or Edison storage battery has an alkaline electrolyte of 21 per cent caustic potash, KOH, in pure water, H_2O. The positive plate when charged consists of nickel hydroxide, $Ni(OH)_2$, held in steel tubes, while the negative plate consists of iron oxide, FeO, held in steel pockets. The plates are separated by rubber insulators and the assembly is mounted in a nickel-plated steel container.

The condition of discharge can best be determined by measuring the terminal voltage, which varies from about 1.4 to 1.0 volts with an average of 1.2 volts at normal discharge rates. The specific gravity is not a reliable indicator of the state of discharge because the electrolyte acts only as a catalytic agent and does not undergo any permanent change during charging or discharging.

As compared to the lead storage battery, the nickel-iron storage battery

1. Is lighter per ampere-hour capacity,
2. Has higher internal resistance with lower efficiency,
3. Is more rugged in construction,
4. Can remain uncharged without injury,
5. Requires more water because it gasses continuously during charge, and evaporates more water, owing to operating at a higher temperature,
6. Must be insulated because the container is metal,
7. Has a limited efficient temperature range.

2.7. Electric Resistance. The Ohm. When a conductor has a large number of free electrons and can easily pass a large current, it is said to

have a low resistance. Conversely, in an insulator with relatively few electrons, high voltages may cause only a very few electrons to flow because of the high resistance of the insulator.

When electrons drift along a conductor, they bump into atoms, freeing other electrons that in turn drift on. These collisions between free elec-

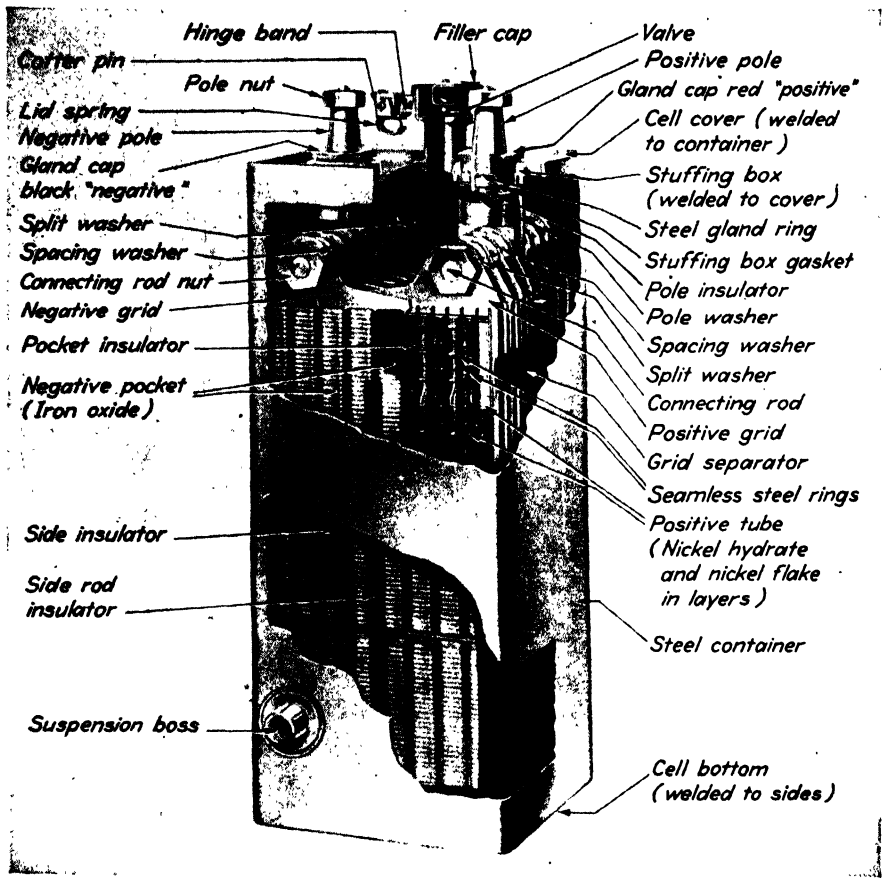
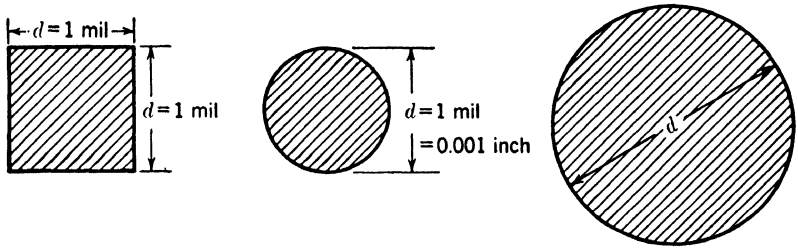


FIG. 2.7. Nickel-iron storage battery showing details.

trons and atoms tend to retard the movement by friction, which acts as a resistance to the flow of electric current. Whenever current flows through a resistance, energy is lost in the form of heat. The current always flows from the higher to the lower potential as already shown in Fig. 2.4. If the potential difference is zero, the current is zero, and if the potential difference is reversed, the current must reverse. *A resistor can be defined as a device in which the flow of electric current always produces heat and nothing else.*

The practical unit for measuring resistance is the *ohm*. *A resistance is said to have one ohm when an emf of one volt causes a current of one ampere to flow.* This is a statement of Ohm's law, which will be studied in more detail later. *The international ohm has been defined as the resistance at* 0°C *of a column of mercury with a uniform cross section,* 106.3 *cm long and weighing* 14.452 *grams.*

2.8. Resistivity. Since resistance is the opposition to electron flow caused by the collision of free electrons with the atoms in the conductor, it



(a) 1 square mil area (b) 1 circular mil area (c) d^2 circular mils area

FIG. 2.8. Illustration of circular mils area = d^2. (Values taken from Smithsonian Physical Tables.)

is reasonable to believe that if the area of the conductor is increased or the length decreased, the resistance will be reduced. The resistance will also depend upon the material in the conductor. Expressing this in equation form

$$R = \frac{\rho l}{A} \tag{2.5}$$

where R = resistance of the conductor, ohms
l = length of the conductor, cm
A = area of the conductor, sq cm
ρ = resistivity, measured in ohms per centimeter cube

In other words the resistivity of a conductor is numerically equal to the resistance between the opposite faces of a centimeter cube.

Resistance per Circular Mil-foot. Since it is common engineering practice to express the length of conductors in feet and the diameter in mils ($\frac{1}{1000}$ in.), the area can be expressed in *circular mils.* The area of a circle is proportional to the diameter squared, so if the area of a circle 1 mil in diameter is 1 circular mil, then the area of any circle measured in circular mils is equal to the square of the diameter in mils (see Fig. 2.8). *A mil-foot of conductor can be defined as having a length of one foot and a cross-section area of one circular mil.* With this terminology, Eq. (2.5) can be rewritten

$$R = \frac{\rho l}{d^2} \tag{2.6}$$

where R = resistance of the conductor, ohms
l = length of the conductor, ft
d = diameter of the conductor, mils
ρ = resistance of conductor, ohms/mil-ft

ρ depends upon the conductor material and the temperature of the conductor. Table 2.2 gives the values of the resistivities for a number of commonly used materials.

TABLE 2.2. RESISTIVITIES AND COEFFICIENTS OF RESISTANCE OF CONDUCTOR MATERIALS*

Material	ρ, ohms per circular mil-foot at 20°C	α, ohms per centigrade degree at 20°C
Aluminum	17	0.0039
Antimony	251	0.0036
Bismuth	722	0.004
Brass	42	0.002
Constantan (40% Ni, 60% Cu)	295	0.000008
Copper, annealed	10.37	0.00393
Copper, hard-drawn	10.65	0.00382
German silver (18% Ni)	199	0.0004
Iron, cast	54	0.005
Lead	132	0.0041
Magnesium	28	0.004
Manganin (84% Cu, 12% Mn, 4% Ni)	265	0.000006
Mercury	577	0.00089
Molybdenum	34	0.0034
Monel metal	253	0.002
Nichrome	602	0.0004
Nickel	47	0.006
Platinum	60	0.003
Silver (99.98% pure)	9.9	0.0038
Steel, annealed sheet	66–300	0.0042
Tin	69	0.0042
Tungsten	33	0.0045
Zinc	36	0.0037

* Values taken from Smithsonian Physical Tables.

Example 2.5. What is the cross-sectional area of a wire 1 mil in diameter?
Solution. The area of a circle is

$$A = \pi r^2 = \frac{\pi d^2}{4}$$

Hence when the diameter is 1 mil, the area is

$$A = \frac{\pi 1^2}{4} = 0.7854 \text{ sq mils area} \qquad \textit{Ans.}$$

Example 2.6. What is the diameter of a 144-circular-mil wire?

Solution

$$d^2 = 144$$
$$d = 12 \text{ mils diameter} \qquad \textit{Ans.}$$

Example 2.7. What is the resistance of 10 miles of aluminum wire ⅛ in. in diameter?
Solution. Taking the value of ρ from Table 2.2 and substituting in Eq. (2.6)

$$R = \frac{17[10(5{,}280)]}{125^2} = 57.44 \text{ ohms} \qquad \textit{Ans.}$$

Example 2.8. What length of 2-mil manganin wire will be required for a 25-ohm standard resistor?
Solution. The resistance of 1 ft by Eq. (2.6) is

$$R = \frac{265(1)}{2^2} = 66.25 \text{ ohms}$$

then

$$\frac{25}{66.25}(12) = 4.53 \text{ in.} \qquad \textit{Ans.}$$

2.9. Temperature Coefficient of Resistance. The resistance of the conductor depends upon the temperature, and it can be expressed as a function of the temperature to a very good approximation by the following formula:

$$R_t = a_0 + a_1 t + a_2 t^2 \tag{2.7}$$

where R_t = resistance at the given temperature t
a_0 = resistance at 0°C
a_1 = first-order temperature increment
a_2 = second-order temperature increment
t = temperature, °C

Almost any quantity that is a continuous function of another quantity can be represented as a power series in terms of the second quantity. The above formula can be regarded as the first three terms of such a power series.

In Fig. 2.9, the solid line is the value of R_t as the temperature of 1 mil-ft of annealed copper wire varies. $a_0 = 9.554$ ohms is the resistance at 0°C. $a_1 = 9.554/234.5 = 0.04074$ is the slope of the curve at 0°C. In other words, the dotted line is tangent to the curve at this point and shows that throughout the usual range of operating temperatures, copper behaves as though its resistance would be zero at −234.5°C. Actually the curve bends up and approaches zero temperature near −273°C. By selecting another point on the curve, such as absolute zero, the second-order constant a_2 can be determined by substituting in Eq. (2.7); thus,

$$0 = 9.554 + (0.04074)(-273) + a_2(-273)^2$$

or

$$a_2 = 0.0000209$$

This will make the curve approach the solid line in Fig. 2.9. For some materials that have a very limited linear section of the resistance-temperature curve, it is desirable to fit the curve with some higher order

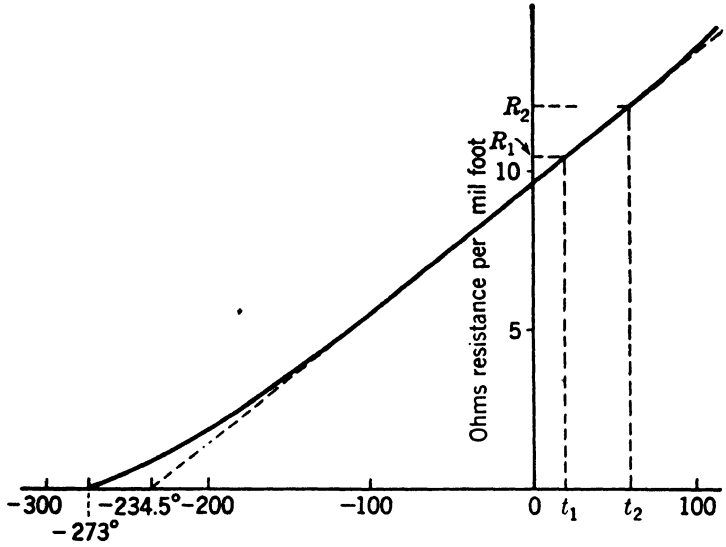


FIG. 2.9. Resistance-temperature curve for annealed copper wire.

terms. For 1 mil-ft of copper wire, the resistance to a second-order approximation is

$$R_t = 9.554 + 0.04074t + 0.0000209t^2 \tag{2.8}$$

For most materials the first-order approximation is sufficiently accurate, so that Eq. (2.7) reduces to

$$R_t = R_0(1 + \alpha_0 t) \tag{2.9}$$

where R_t = resistance of conductor at temperature t, ohms
R_0 = resistance of conductor at 0°C, ohms

$$\alpha_0 = \frac{a_1}{a_0} = \text{the temperature coefficient at 0°C}$$

For annealed copper wire, $\alpha_0 = 0.04074/9.554 = 0.00427$, which means that a copper wire having 1-ohm resistance at 0°C will have a resistance of 1.00427 ohms at 1°C.

In engineering it is customary to give the resistivity and temperature coefficients for 20°C, which is 68°F, an average room temperature. This can be done readily by using the first approximation of Taylor's series which for this case can be written

$$R_2 = b_1 + b_2(t_2 - t_1) \tag{2.10}$$

where R_2 = resistance at temperature t_2, ohms
b_1 = resistance at temperature t_1, ohms
b_2 = first-order temperature increment

Now for the case of 1 mil-ft of annealed copper wire at 20°C, by Eq. (2.9)

$$b_1 = R_{20} = 9.554[1 + 0.00427(20)] = 10.37 \text{ ohms}$$

Now in Eq. (2.10), when $R_2 = 9.554$, $t_2 = 0$, as shown in Fig. 2.9. Making these substitutions

$$9.554 = 10.37 + b_2(0 - 20)$$

or

$$b_2 = 0.04074$$

which is the slope at 20°C, which is the same as it was at 0°C. Equation (2.9) can now be written

$$R_2 = R_1[1 + \alpha_1(t_2 - t_1)] \tag{2.11}$$

where R_2 = resistance at temperature t_2, ohms
R_1 = resistance at temperature t_1, ohms
$\alpha_1 = \dfrac{b_2}{b_1}$ = the temperature coefficient at temperature t_1

For 1 mil-ft of annealed copper wire at 20°C

$$\alpha_1 = \frac{b_2}{b_1} = \frac{0.0408}{10.37} = 0.00393$$

The resistance of 1 mil-ft at some other temperature t_2 is then

$$R_2 = 10.37[1 + 0.00393(t_2 - 20)] \tag{2.12}$$

The values of $R_1 = 10.37$ and $\alpha_1 = 0.00393$ are given in Table 2.2 for annealed copper wire. The resistance of any of the other materials at some temperature t_2 can be found in a similar fashion. It should be noted that the temperature coefficient is different for different temperatures, being 0.00427 at 0°C and 0.00393 at 20°C; hence one must *always use the temperature coefficient corresponding to the starting temperature used.* Referring to Table 2.2, it will be noted that many of the temperature coefficients are about 0.004 for a starting temperature of 20°C. This is a convenient figure to remember.

Example 2.9. A coil of copper wire has a resistance of 100 ohms at 10°C. What is its resistance at 80°C?

Solution. By Eq. (2.11) the resistance at 20°C will be

$$100 = R_1[1 + 0.00393(10 - 20)]$$

$$R_1 = \frac{100}{0.9607} = 104.1 \text{ ohms at } 20°\text{C}$$

Then at 80°C Eq. (2.11) gives

$$R_2 = 104.1[1 + 0.00393(80 - 20)] = 128.6 \text{ ohms} \qquad \textit{Ans.}$$

Example 2.10. The cold (20°C) resistance of a transformer winding measures 4.35 ohms. The hot resistance measures 5.81 ohms. What is the temperature rise?

Solution. Assuming that copper wire was used in the winding, Eq. (2.11) gives

TABLE 2.3. COPPER-WIRE TABLE, STANDARD ANNEALED COPPER
American Wire Gauge (B & S)

Gauge number	Diameter, mils at 20°C	Cross section at 20°C		Ohms per 1,000 ft at 20°C (= 68°F)	Pounds per 1,000 ft	Feet per pound	Feet per ohm at 20°C (= 68°F)	Ohms per pound at 20°C (= 68°F)	Gauge number
		Circular mils	Sq in.						
0000	460 0	211,600.0	0.1662	0 04901	640.5	1.561	20,400.0	0.00007652	0000
000	409 6	167,800.0	0.1318	0 06180	507.9	1.968	16,180.0	0 0001217	000
00	364.8	133,100.0	0.1045	0 07793	402.8	2.482	12,830.0	0.0001935	00
0	324.9	105,500 0	0.08289	0 09827	319 5	3.130	10,180.0	0 0003076	0
1	289.3	83,690.0	0.06573	0 1239	253 3	3.947	8,070 0	0 0004891	1
2	257.6	66,370.0	0 05213	0 1563	200 9	4.977	6,400 0	0.0007778	2
3	229 4	52,640 0	0 04134	0 1970	159 3	6.276	5,075 0	0 001237	3
4	204 3	41,740 0	0 03278	0 2485	126.4	7.914	4,025.0	0.001966	4
5	181 9	33,100 0	0.02600	0 3133	100.2	9.980	3,192.0	0.003127	5
6	162 0	26,250 0	0 02062	0 3951	79 46	12 58	2,531 0	0 004972	6
7	144 3	20,820 0	0 01635	0.4982	63 02	15.87	2,007.0	0.007905	7
8	128.5	16,510 0	0 01297	0 6282	49 98	20.01	1,592.0	0 01257	8
9	114 4	13,090 0	0 01028	0 7921	39 63	25 23	1,262 0	0 01999	9
10	101.9	10,380 0	0 008155	0 9989	31 43	31.82	1,001 0	0 03178	10
11	90 74	8,234 0	0 006467	1 260	24 92	40.12	794 0	0 05053	11
12	80 81	6,530 0	0 005129	1 588	19.77	50.59	629 6	0 08035	12
13	71 96	5,178 0	0 004067	2 003	15 68	63 80	499 3	0.1278	13
14	64.08	4,107 0	0 003225	2 525	12 43	80.44	396 0	0.2032	14
15	57 07	3,257 0	0.002558	3.184	9.858	101.4	314.0	0 3230	15
16	50 82	2,583.0	0 002028	4 016	7 818	127.9	249.0	0 5136	16
17	45 26	2,048.0	0.001609	5 064	6 200	161.3	197.5	0 8167	17
18	40 30	1,624 0	0 001276	6 385	4 917	203 4	156.6	1 299	18
19	35 89	1,288 0	0 001012	8.051	3 899	256 5	124 2	2.065	19
20	31 96	1,022.0	0.0008023	10 15	3 092	323 4	98 50	3.283	20
21	28 46	810 1	0.0006363	12 80	2.452	407 8	78 11	5 221	21
22	25 35	642.4	0.0005046	16.14	1 945	514.2	61 95	8.301	22
23	22.57	509 5	0.0004002	20.36	1 542	648.4	49.13	13.20	23
24	20 10	404 0	0.0003173	25 67	1 223	817.7	38 96	20.99	24
25	17 90	320.4	0.0002517	32.37	0 9699	1,031.0	30.90	33 37	25
26	15 94	254.1	0.0001996	40.81	0 7692	1,300.0	24.50	53.06	26
27	14 20	201.5	0.0001583	51.47	0 6100	1,639 0	19.43	84.37	27
28	12.64	159 8	0 0001255	64.90	0.4837	2,067.0	15.41	134 2	28
29	11.26	126.7	0.00009953	81.83	0.3836	2,607.0	12 22	213.3	29
30	10.03	100 5	0.00007894	103.2	0 3042	3,287.0	9.691	339.2	30
31	8.928	79.70	0.00006260	130.1	0 2413	4,145.0	7.685	539.3	31
32	7.950	63.21	0.00004964	164.1	0 1913	5,227.0	6.095	857.6	32
33	7.080	50 13	0.00003937	206.9	0.1517	6,591.0	4.833	1,364.0	33
34	6.305	39.75	0.00003122	260.9	0.1203	8,310.0	3.833	2,168.0	34
35	5.615	31.52	0.00002476	329.0	0.09542	10,480.0	3.040	3,448.0	35
36	5.000	25.00	0.00001964	414.8	0.07568	13,210.0	2.411	5,482.0	36
37	4.453	19.83	0.00001557	523.1	0.06001	16,660.0	1.912	8,717.0	37
38	3.965	15.72	0.00001235	659.6	0.04759	21,010.0	1.516	13,860.0	38
39	3.531	12.47	0.000009793	831.8	0.03774	26,500.0	1 202	22,040.0	39
40	3.145	9.888	0.000007766	1,049.0	0.02993	33,410.0	0.9534	35,040.0	40

$$5.81 = 4.35[1 + 0.00393(t_2 - t_1)]$$
$$t_2 - t_1 = 85.5° = \text{the temperature rise} \qquad \textit{Ans.}$$

Example 2.11. What is the resistance of 10 ft of No. 40 nichrome resistance wire at a temperature of 100°C?

Solution. From Table 2.3 the diameter is $d = 3.145$ mils. Then by Eq. (2.6) and Table 2.2, the resistance of 10 ft at 20°C is

$$R = \frac{602(10)}{(3.145)^2} = 608.6 \text{ ohms}$$

By Eq. (2.11) the resistance at 100°C is

$$R = 608.6[1 + 0.0004(100 - 20)] = 628.1 \text{ ohms} \qquad \textit{Ans.}$$

2.10. Electric Energy. The Joule. The purpose of all electrical systems is to transport energy. The power plants generate large amounts of electric energy, which is transported over power transmission lines and used by the consumer when it is converted into light or heat by resistors or into mechanical power by electric motors. Telephone lines carry electric energy that corresponds to the acoustic energy received by the telephone transmitter and deliver this energy to the telephone receiver, where it is again converted into an almost identical form of acoustical energy. The radio-broadcast station sends energy through space to a distant radio-receiving antenna. Although only a small amount of energy is received, it is enough to produce signals in the radio receiver that are clearly understood.

From the definition of the volt, it takes *one joule of energy to move one coulomb through a potential difference of one volt.* The general equation is then

$$W = QE \tag{2.13}$$

where W = energy, joules
Q = charge, coulombs
E = potential difference, volts

This charge Q can be moved through the potential difference E in 1 sec if 1 amp flows, or in 1,000 sec if only 1 ma flows. This relationship of the charge with respect to current and time can be stated mathematically as

$$Q = It \tag{2.14}$$

where Q = charge, coulombs
I = current, amp
t = time, sec

Substituting Q of Eq. (2.14) into Eq. (2.13) gives

$$W = EIt \tag{2.15}$$

In words, this equation says that energy can be expressed as directly

equal to the product of voltage, current, and time. If the voltage and current are not steady, the energy can be obtained by integrating over the period of time; thus

$$W = \int EI\, dt \tag{2.16}$$

Electric energy is commonly measured by a watt-hour meter, which integrates EI over time as indicated by this equation.

The joule as defined here is an electrical quantity. Actually, the joule was first defined as a mechanical quantity, in connection with studies in the mechanical equivalent of heat. Heat, however produced, requires the expenditure of a certain amount of mechanical energy. If it is produced in an electric circuit, it can be regarded as derived from the mechanical energy required to turn the dynamo or generator producing the current. This shows that an electric circuit is a means of transporting mechanical energy.

Example 2.12. An electric heater operating on a 115-volt line draws 15 amp. How many joules of electric energy are converted into heat in 1 hr?

Solution. By Eq. (2.15)

$$W = (115)(15)(60) = 103{,}500 \text{ joules} \qquad \textit{Ans.}$$

2.11. Electric Power. The Watt. *Power can be defined as the time rate of expending or storing energy.* Power is measured in *watts*. Power in watts is numerically equal to voltage in volts multiplied by current in amperes; thus

$$P = EI \tag{2.17}$$

where P = power, watts
E = voltage, volts
I = current, amp

This equation gives the constant power in a circuit when both the voltage and current are constant.

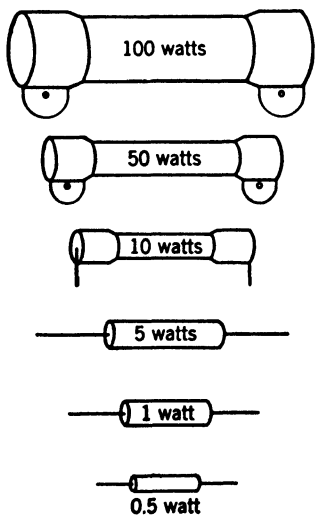
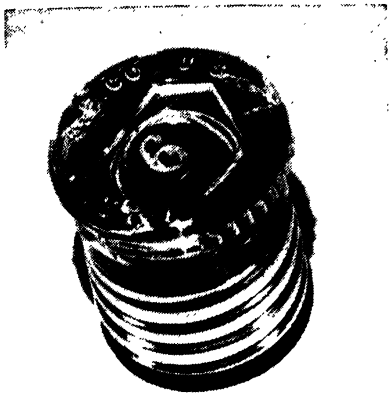
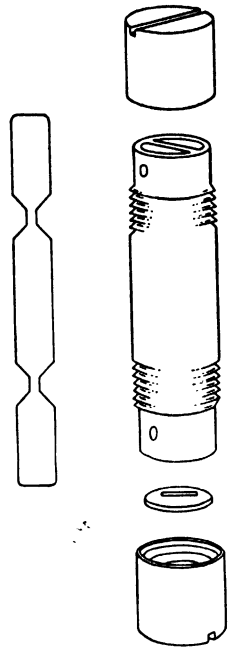


FIG. 2.10. Six sizes of commercial resistors, all of the same resistance value.

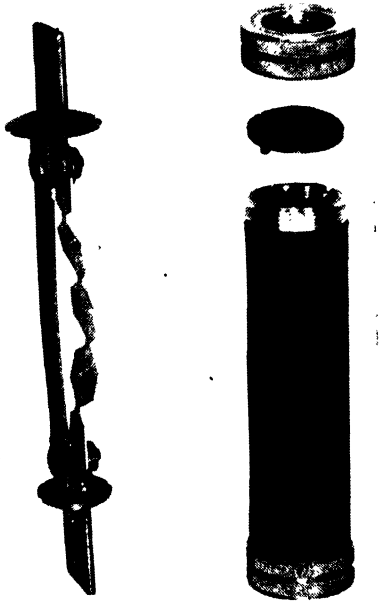
In power circuits the watt is too small to express conveniently the input to electrical-machinery and electric-light loads. For this purpose the *kilowatt*, which is equal to 1,000 watts, is usually used. In communication circuits very small amounts of power are used, so that the *milliwatt*, which is equal to $\frac{1}{1000}$ watt, is often used. The size of resistors for circuit elements are rated in the amount of power they will dissipate as shown in Fig. 2.10.



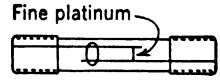
(a) Plug-in type of house fuse up to 30 amp



(b) Refillable cartridge



(c) Knife blade cartridge-type (delay-type link)



(d) Instrument fuse

FIG. 2.11. Typical fuses.

Example 2.13. At 3 cents a kilowatt-hour, what is the cost to operate a 100-watt electric iron 160 hr per month?

Solution

$$\begin{aligned} 100 \text{ watts} &= 0.1 \text{ kw} \\ \text{Electric energy used} &= Pt \\ &= (0.1)(160) = 16 \text{ kwhr} \\ \text{Cost per month} &= (\$0.03)(16) = \$0.48 \end{aligned}$$ *Ans.*

2.12. Circuit Protection. Sources of energy must be protected against overloads that exceed their designed wattage rating. This is usually done by heating or magnetic devices. Only the heating types will be treated now since magnetism is yet to be studied.

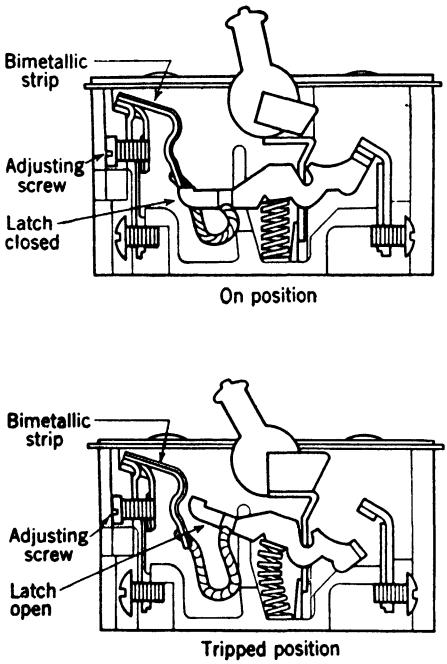


FIG. 2.12. Low-powered thermal circuit breaker.

a. Fuses. The most common and simplest circuit protector is the *fuse*. The fuse material is merely a short length of wire or ribbon made of an alloy that has a very low melting point. The fuse is designed to withstand heating caused by the rated current, but with overload currents the power loss gives sufficient heat to melt the fuse link. The fuse is said to "blow," and thus opens the circuit and protects the equipment. It is important to know fuse characteristics, because some will withstand two or three times their current rating while others will blow at slightly over their rated value. Fuses for commercial applications are usually designed to have a delay action; this is accomplished by making the fuse link of heavy construction except in one or two short portions of its length. This structure allows heat to be carried away from the small high-resistance portion, delaying the melting time, to give the desired delay action (see Fig. 2.11 for typical fuses).

b. Thermal Circuit Breakers. A thermal circuit breaker is a switch that opens the circuit when a current of a limiting value flows for the required length of time. This type of circuit breaker is supplanting the fuse in many applications because it does not use up materials as the fuse does. The circuit breaker is operated by a heater element and bimetallic

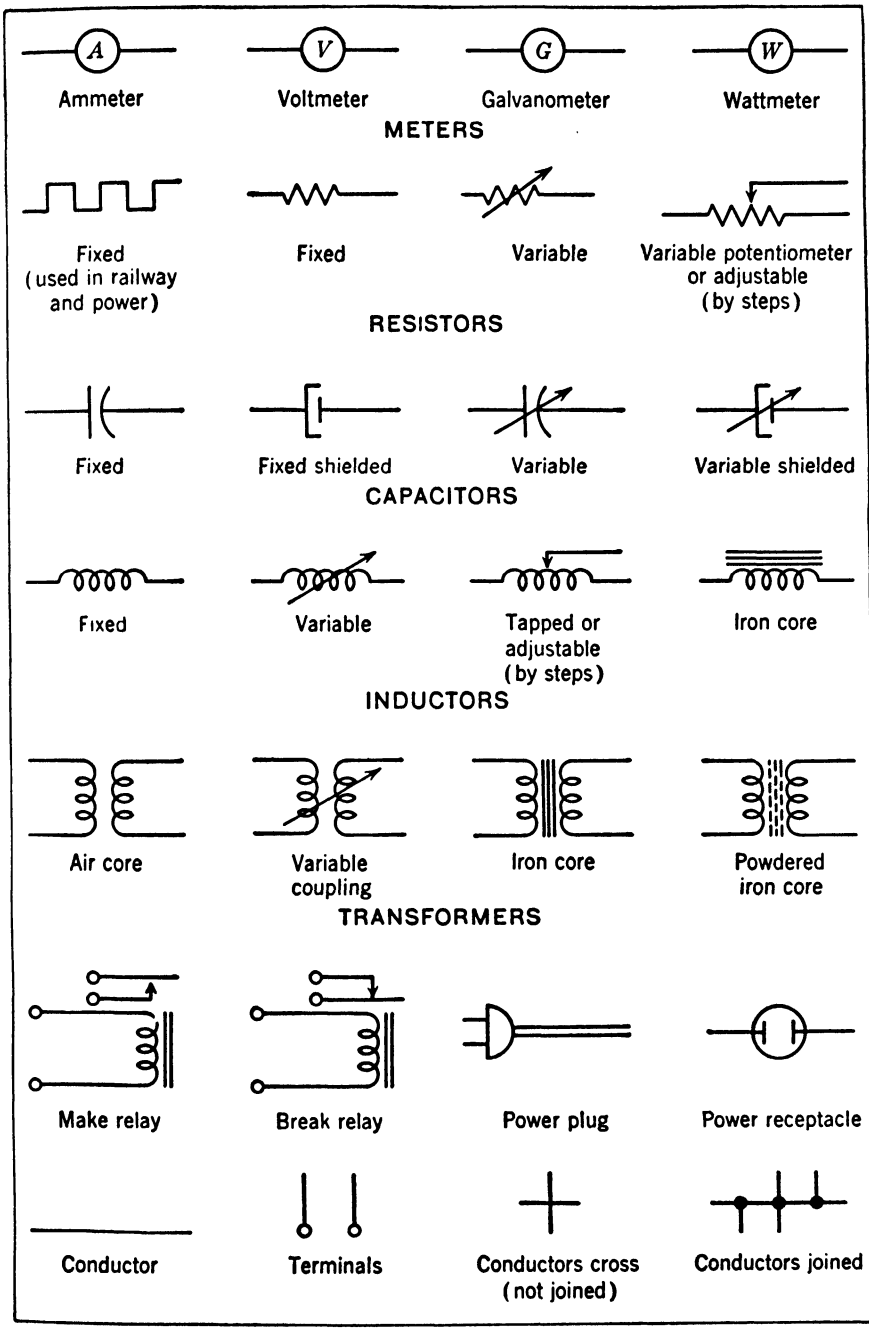


FIG. 2.13. Graphical symbols.

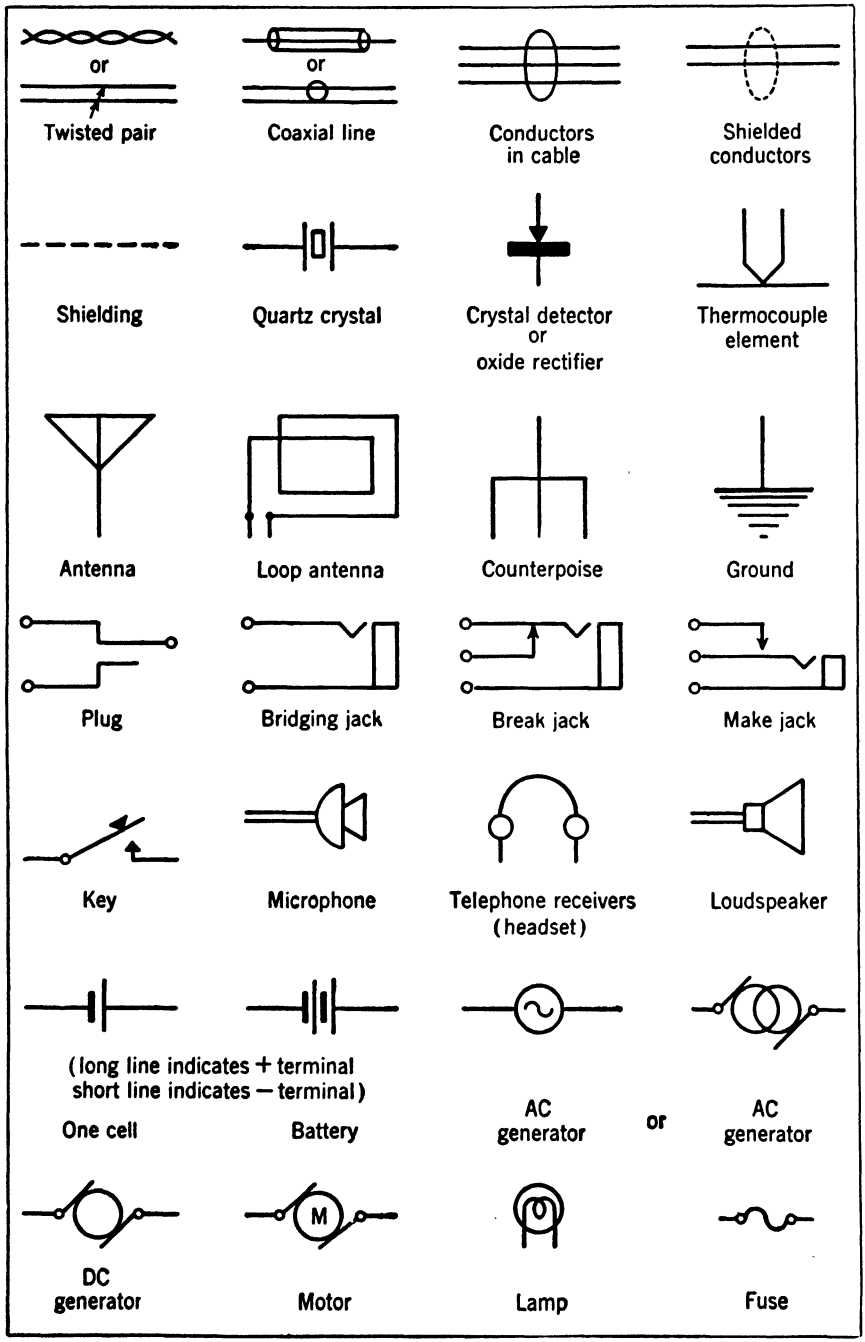


FIG. 2.13. Graphical symbols. (*Continued*).

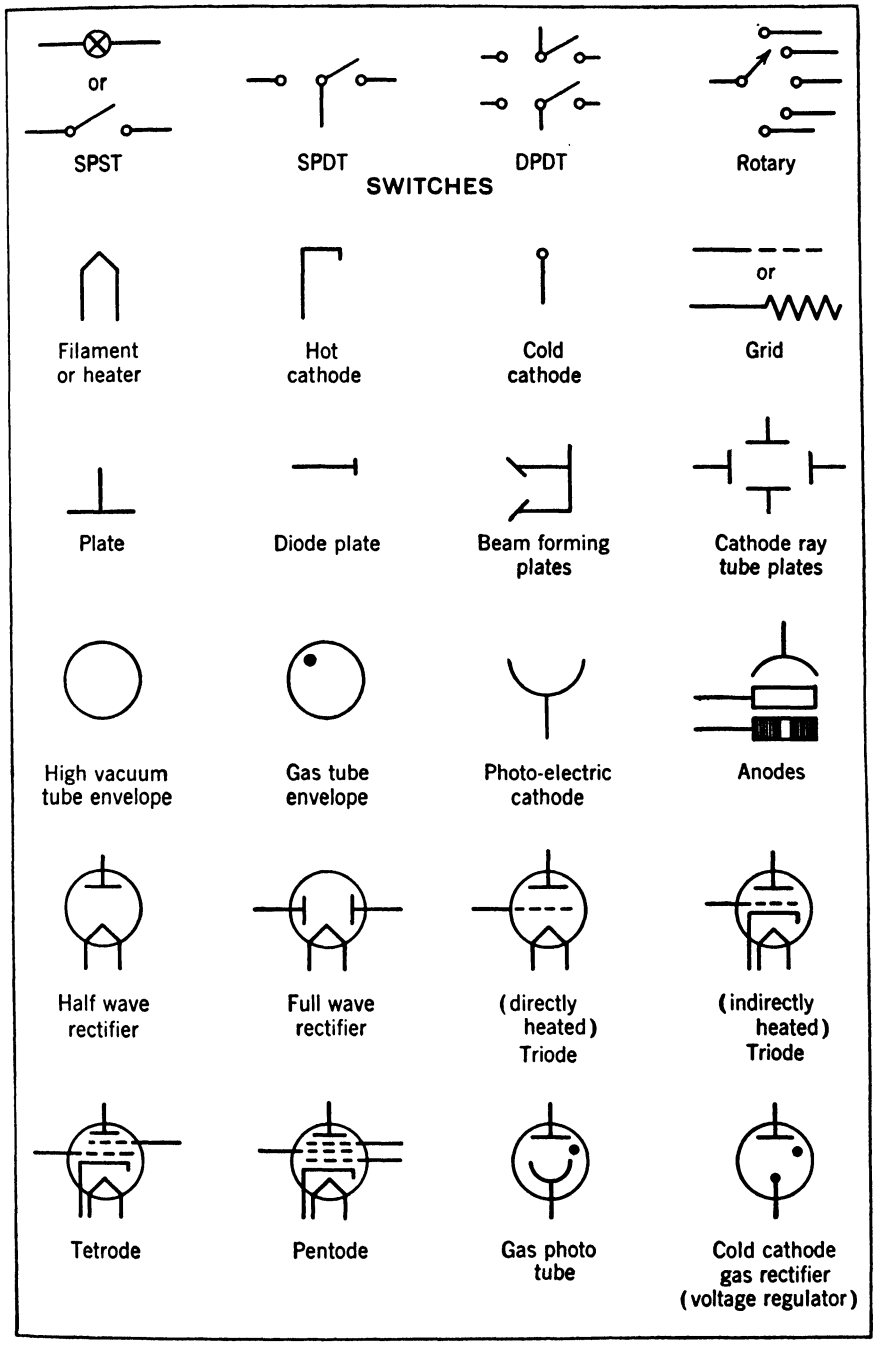


FIG. 2.13. Graphical symbols. (*Continued*).

strip or by passing current through the bimetallic strip itself, which generates the heat. The heat caused by the current makes the bimetallic strip bend, releasing a trip that opens the circuit contactors. Circuit breakers of this type are designed to carry the rated load indefinitely and can be adjusted to carry 50 per cent overload for about 1 min, 100 per cent overload for 20 sec, and a 200 per cent overload for 5 sec. This allows for short overloads that will not injure the equipment. A typical thermal circuit breaker is shown in Fig. 2.12.

c. Heat Coils. When the circuits must be fused closely on steady currents, heat coils are sometimes used by themselves or in conjunction with circuit breakers or fuses. The heat coil consists essentially of a coil of fine wire wound on a copper tube, in which a pin is soldered with a low-melting solder. If the rated current is exceeded, the solder melts, and a spring forces the pin against a ground strip, shorting the circuit and protecting the equipment. Heat coils are quite commonly used in telephone central offices to protect the apparatus.

2.13. Schematic Symbols. The use of schematic symbols simplifies the appearance of complicated circuits. By means of a standard set of symbols, circuit elements can be presented clearly in a uniform graphical sign language. For reference purposes these symbols are presented in Fig. 2.13.

Summary

Coulomb's law gives the force between two concentrated electric charges as

$$F = \frac{Q_1 Q_2}{K d^2}$$

where F = force, dynes
Q_1 and Q_2 = charge on respective bodies, coulombs
d = distance between charges, cm
K = dielectric constant ($K = 1$ for air)

1 coulomb charge = 6.289×10^{18} electrons charge

Electric current, the motion of charges, is measured in amperes.

$$1 \text{ amp} = 1 \text{ coulomb/sec} = 6.289 \times 10^{18} \text{ electrons/sec}$$

Electric current density is measured in amperes per unit area and can be expressed by

$$i = nve$$

where i = current density, amp/sq cm
n = free electrons/cu cm
v = average electron velocity, cm/sec
$e = 1.59 \times 10^{-19}$ coulombs/electron

The international ampere is equal to a steady current that will deposit 0.001118 g of silver per second when passed through a specified silver nitrate solution.

Electron current flows from negative to positive in an external circuit.

Conventional current flows from positive to negative in an external circuit.

Electric currents are either direct or alternating. Direct currents (d-c) always flow

in the same direction. Alternating currents (a-c) flow first in one direction and then in the other direction.

Potential difference is an electric difference pressure measured in volts that causes (or tends to cause) an electric current to flow.

The voltage between two points is numerically equal to the joules of energy gained or lost in moving 1 coulomb of charge between the two points. One volt is the pressure required to cause 1 amp to flow through 1 ohm.

Electromotive force (emf) is measured in volts and is the work done by a generator on each coulomb charge passing through it.

Electric batteries are devices to convert chemical energy into electrical energy. They consist of a positive and negative plate of unlike conductors in an electrolyte that acts chemically upon one of the plates.

A primary cell uses up the original materials and is thrown away after its useful life is accomplished. Dry cells are the common commercial form of the primary cell.

A storage cell uses up original materials in discharging but replaces them by electrolytic action when charging. When the cell is charged, the stored energy is chemical, not electrical. The common commercial forms of the storage cell are used in lead storage batteries and nickel-iron storage batteries.

Lead storage batteries consist of lead peroxide positive plates and spongy lead negative plates in a dilute solution of sulfuric acid. In discharging, both positive and negative plates change to lead sulfate and the electrolyte changes from sulfuric acid to water. In charging, the reverse action takes place.

Nickel-iron storage batteries consist of nickel hydroxide positive plates and iron oxide negative plates in a 21 per cent caustic potash solution. In discharging, the nickel hydroxide changes to a lower oxide, the iron changes to iron hydroxide, and the electrolyte acts as a catalytic agent.

Resistance to current flow is measured in ohms. Resistance always produces heat and nothing else. The international ohm is the resistance of a column of mercury of uniform cross section, 106.3 cm long and weighing 14.452 g.

Resistivity measured in *ohm-centimeters* is the resistance between parallel faces of a centimeter cube, or measured in *ohms per mil-foot* is the resistance of a conductor 1 ft long with a cross-section area of 1 circular mil.

The *resistance* of a conductor is given by

$$R = \frac{\rho l}{d^2}$$

where R = conductor resistance, ohms
l = conductor length, ft
d = conductor diameter, mils
ρ = conductor resistivity at a specified temperature, ohms per mil-ft

The temperature coefficient of resistance, at the starting temperature, is the resistance change per degree per ohm.

The effect of temperature on the resistance of a conductor can be computed from the formula

$$R_2 = R_1[1 + \alpha_1(t_2 - t_1)]$$

where R_2 = resistance at final temperature t_2, ohms
R_1 = resistance at starting temperature t_1, ohms
α_1 = temperature coefficient at temperature t_1

For *copper* with an initial temperature of 20°C, this equation reduces to

$$R_2 = R_1[1 + 0.00393(t_2 - 20)]$$

For most pure metals the *temperature coefficient is about* 0.004. For alloys it is usually less and may become negative. For insulators it is always negative and comparatively large.

Direct-current *energy* is measured in *watt-seconds* and can be computed from

$$W = EQ, \qquad EIt = \int EI\,dt = \int p\,dt$$

where W = energy, joules
E = potential difference, volts
Q = charge, coulombs
I = current, amp
p = instantaneous power, watts
t = time, sec

Direct-current *power* is measured in *watts* as given by

$$P = EI$$

where P = power, watts
E = potential difference, volts
I = current, amp

Electric circuits are *protected* by *heating* or *magnetic devices.*

Fuses are portions of the electric circuit made of low-melting-point alloy that will melt and open the circuit on overload currents.

Thermal circuit breakers are switches actuated by a bimetallic strip that bends when heated.

Heat coils melt solder holding a pin that is forced by a spring to close a circuit.

Schematic symbols are a form of graphic sign language used to simplify and clarify complicated circuits.

Exercises

2.1. Two equal negative charges are separated in air by 10 cm. If the force between the charges is 0.5 dyne, how many electrons are on each charge?

2.2. How long will it take 20 amp to deposit 1 g of silver on an electrode in a standard silver nitrate solution?

2.3. What is the current in a No. 14 copper conductor when the electron drift is 0.1 cm/sec?

2.4. A wet cell produces an emf of 1 volt. What electrodes were probably used?

2.5. What is the chemical reaction in a lead storage battery on charging and discharging?

2.6. Hard-drawn copper wire has a diameter of 35.89 mils. How many feet will be required to make 10 ohms of resistance?

2.7. A resistance of 100 ohms at room temperature increases to 106 ohms at 40°C. What is the temperature coefficient and of what kind of metal is the resistor probably made?

2.8. A pulse of energy lasting for 1 μsec is created by 10,000 volts at 2 amp. What is the energy in the pulse in joules?

2.9. What is the maximum current that can be used in a 50-watt, 1,500-ohm resistor?

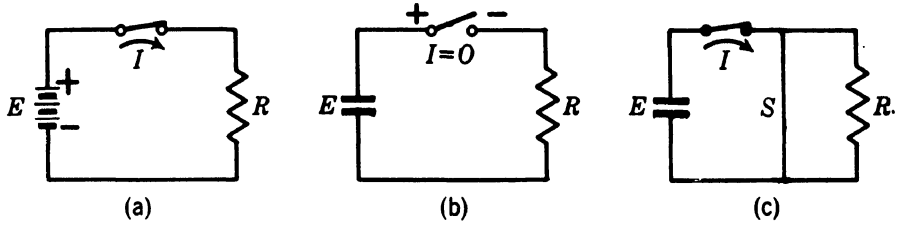
2.10. How is delay action obtained in fuses?

CHAPTER 3

DIRECT-CURRENT CIRCUITS

The electrical quantities discussed in the last chapter will now be used in simple d-c circuits. Series, parallel, and series-parallel circuits will be treated in this chapter, leaving the more complicated circuits to be treated in the next chapter.

3.1. Simple Electric Circuit. The simplest form of an electric circuit consists of a source of emf, the terminals of which are connected by a continuous conducting path through a resistor. A simple closed d-c



(a) Closed circuit (b) Open circuit (c) Short circuit

FIG. 3.1. Simple electric circuits.

circuit is illustrated in Fig. 3.1*a*. In this circuit the source of emf is a cell, the terminals of which maintain a constant voltage E of unchanging polarity across the resistor R. The voltage E causes the current I to flow in the circuit through the closed switch and the resistance R. If the switch makes good contact, its resistance can be neglected; however if the contact is loose and dirty, the resistance may increase so much at this point as to make the circuit inoperative.

If the switch is opened as shown in Fig. 3.1*b*, the current will stop flowing, and the battery voltage will appear across the contacts of the open switch. This is a common method, used intentionally, to interrupt the flow of current. When the switch is opened, the current ceases to flow, and the energy of the battery can thus be conserved until it is needed. On the other hand if the conductor becomes broken, the defect is known as an *open circuit.*

If the switch is closed and an excessive current flows from the battery but does not reach the load resistor, the circuit is said to have a short circuit as illustrated in Fig. 3.1c. In this case the current flows through the short-circuiting wire S. The short-circuit current is limited by the internal resistance of the battery. For example, a good No. 6 dry cell will draw about 30 amp on short circuit. Short circuits are seldom

intentional, because in most cases they do not serve a useful purpose; they are usually the result of a defect in the wiring. For example, if the insulation is worn off a twisted pair of wires, they may become short-circuited.

3.2. Ohm's Law. A German physicist, Georg Ohm, was the first to discover the relation between the voltage, current, and resistance in an electric circuit. Since voltage and resistance are usually the independent values or parameters, the most common form of Ohm's law can be stated as follows: *The current flowing in a circuit is directly proportional to the voltage across the circuit and inversely proportional to the resistance of the circuit.* In equation form

$$I = \frac{E}{R} \tag{3.1}$$

where I = current, amp
E = voltage, volts
R = resistance, ohms

This law can be stated in two other ways as follows:

The voltage across a circuit is directly proportional to the current flowing in the circuit and to the resistance of the circuit. Thus

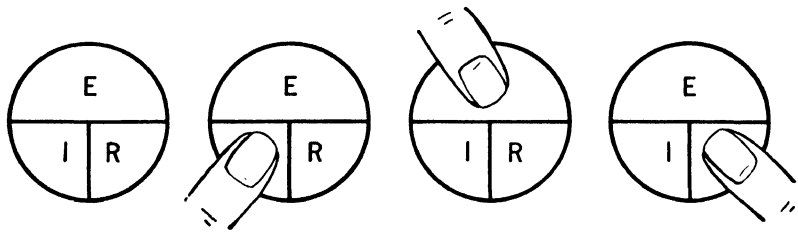
$$E = IR \tag{3.2}$$

where the values are as defined in Eq. (3.1).

Or *the resistance of a circuit is directly proportional to the voltage across the circuit and inversely proportional to the current flowing in the circuit.* Hence, mathematically,

$$R = \frac{E}{I} \tag{3.3}$$

where the values are as defined in Eq. (3.1).



(a) Chart (b) $I = E/R$ (c) $E = IR$ (d) $R = E/I$

FIG. 3.2. Instructional aid chart for Ohm's law.

Ohm's law is one of the simplest and most important laws in all electrical work. A popular way to remember it is to place the letters in a circle as shown in Fig. 3.2*a*. When the quantity desired is covered, the other quantities form the right-hand side of the equation as shown in Figs. 3.2*b*, *c*, and *d* for the three forms of the law.

An important principle to remember in the use of Ohm's law is that the law can be applied to any portion of a circuit. If it is applied to an entire circuit, then the current flowing in the entire circuit is directly proportional to the total voltage across the entire circuit and inversely proportional to the total resistance in the entire circuit. If the law is applied to a part or branch of a circuit, then the current flowing in the branch is directly proportional to the voltage across the branch and inversely proportional to the resistance of the branch. The important thing to remember is that the current, voltage, and resistance values must deal with the same part of the circuit.

Example 3.1. What is the current in a 15-ohm resistor when the emf across it is 6 volts?

Solution. By Ohm's law as given in Eq. (3.1)

$$I = \frac{E}{R} = \frac{6}{15} = 0.4 \text{ amp} \qquad \textit{Ans.}$$

Example 3.2. What is the voltage across a 2,300-ohm resistor carrying 16 ma?

Solution. By Ohm's law as given in Eq. (3.2)

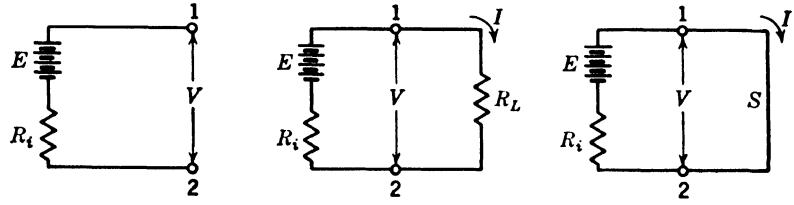
$$E = IR = (0.016)(2{,}300) = 36.8 \text{ volts} \qquad \textit{Ans.}$$

Example 3.3. Through what resistance will 50 mv produce a current of 75 μamp?

Solution. By Ohm's law as given in Eq. (3.3)

$$R = \frac{E}{I} = \frac{0.05}{75 \times 10^{-6}} = 667 \text{ ohms} \qquad \textit{Ans.}$$

3.3. Terminal Voltage. A voltage source in general consists of an internal voltage E and an internal resistance R_i as shown in Fig. 3.3.



(a) Open circuit $V = E$ (b) Closed circuit $V = E - IR_i$ (c) Short circuit $V = 0$

FIG. 3.3. Battery terminal voltages.

The terminal voltage of such a battery or generator is measured between its output terminals. In Fig. 3.3 the battery terminal voltage is measured between terminals 1 and 2. The terminal voltage is equal to the *internal voltage* when the terminals are open-circuited, as shown in Fig. 3.3*a*. In this condition no current flows, hence the voltage drop across R_i is zero, and the total internal voltage E appears across the output terminals 1-2.

When a load resistance R_L is connected across the terminals of a battery as shown in Fig. 3.3*b*, a current I will flow in the circuit. The current I flowing through the internal resistance R_i will drop the terminal voltage below the internal voltage. For this case, the value of the terminal voltage is

$$V = E - IR_i \tag{3.4}$$

where V = terminal voltage, volts
E = internal voltage, volts
I = current, amp
R_i = internal resistance, ohms

For illustration, when the load resistance R_L is equal to the internal resistance R_i, the respective voltage drops will each be equal to one-half the internal voltage. Thus, the terminal voltage for this special case will be one-half the internal voltage of the battery.

Now consider what happens if the terminals of the battery are short-circuited as shown in Fig. 3.3*c*. The terminal voltage must go to zero; hence the voltage drop in the internal resistance of the battery must equal the internal voltage of the battery. Since the internal resistance is usually small, the short-circuit current must be quite large in order that their product by Ohm's law will be equal to the internal voltage of the battery.

Example 3.4. A 1.5-volt dry cell draws 30 amp on short circuit. What is its internal resistance?

Solution. By Eq. (3.4)

$$V = E - IR_i$$
$$0 = 1.5 - 30R_i$$
$$R_i = \frac{1.5}{30} = 0.05 \text{ ohm} \qquad \textit{Ans.}$$

Example 3.5. If the battery of Example 3.4 is connected to a 2-ohm load resistance, what will be the value of the terminal voltage?

Solution. By Ohm's law the current in the complete circuit is

$$I = \frac{E}{R_i + R_L} = \frac{1.5}{0.05 + 2} = 0.732 \text{ amp}$$

By Eq. (3.4)

$$V = E - IR_i$$
$$= 1.5 - (0.732)(0.05)$$
$$= 1.5 - 0.0366 = 1.4634 \text{ volts} \qquad \textit{Ans.}$$

3.4. Simple Series Circuits. When pieces of electrical conductor are connected end to end to form a closed circuit, they are said to be in *series*. In Fig. 3.1*a*, the battery, switch, and resistor are connected in series. Figure 3.4 illustrates two resistors in series with a dry cell. A series circuit may contain any number of pieces of electrical conductor such as

resistors. In fact, as will be shown later, a series circuit may also contain any number of voltage sources.

The fundamental law of a series circuit is that all parts of the circuit contain the same current. To determine this current, add all the resistances to obtain the total resistance. In equation form, the total resistance is

$$R = R_1 + R_2 + R_3 + \cdots + R_n \tag{3.5}$$

where R = total series resistance, ohms
$R_1, R_2, R_3, \ldots, R_n$ = the individual resistors of the series circuit, ohms
n = number of series resistors

With the total resistance and the applied voltage, Ohm's law can be used to determine the current.

(a) Wiring diagram of a series circuit.

(b) Schematic diagram of a series circuit.

FIG. 3.4. Two resistors in series with a dry cell.

Example 3.6. In the series circuit of Fig. 3.4, if $E = 6.6$ volts, $R_1 = 11$ ohms, and $R_2 = 22$ ohms, determine the current.

Solution. By Eq. (3.5) the total series resistance is

$$R = R_1 + R_2 = 11 + 22 = 33 \text{ ohms}$$

By Ohm's law as given in Eq. (3.1)

$$I = \frac{E}{R} = \frac{6.6}{33} = 0.2 \text{ amp} \qquad \textit{Ans.}$$

3.5. Kirchhoff's Voltage Law. Series circuit problems can also be solved by adding the voltage drops across each of the series resistors to get the total voltage. In other words, by Kirchhoff's voltage law, *the sum of the voltage drops around a series circuit is equal to the applied voltage.* In equation form

$$E = IR_1 + IR_2 + IR_3 + \cdots + IR_n \tag{3.6}$$

where E = applied series voltage, volts
I = series current, amp
$R_1, R_2, R_3, \ldots, R_n$ = the individual series resistors, ohms
n = number of series resistors

Example 3.7. Solve Example 3.6 by applying Kirchhoff's voltage law.

Solution. By Eq. (3.6) the applied series voltage is

$$E = IR_1 + IR_2$$

Substituting values

$$6.6 = 11\,I + 22\,I$$
$$= 33I$$

Solving $$I = \frac{6.6}{33} = 0.2 \text{ amp} \qquad \textit{Ans.}$$

3.6. Voltages and Resistances in Series. In a series circuit any number of resistances can be added with the result that the current will be the same in each resistor. This is true of any series circuit. It is also true of voltage with the exception that the polarity will determine the sign. The voltage is said to be higher, or more positive, at the positive end

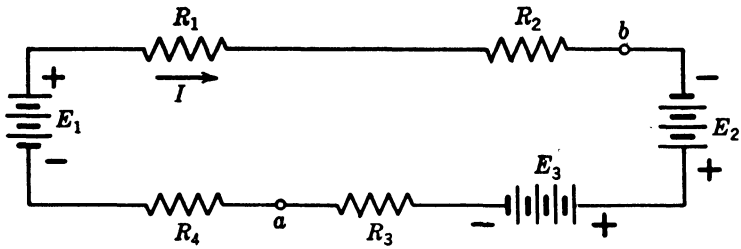


FIG. 3.5. Voltages and resistances in series.

of the battery than at the negative end of the battery. This is spoken of as a *voltage rise.* If all the voltage rises are placed on one side of an equation, then all the voltage drops can be placed on the other side of the equation, because by Kirchhoff's voltage law *the sum of the voltage rises must equal the sum of the voltage drops around the circuit.* In equation form

$$\Sigma \text{ (voltage rises)} = \Sigma \text{ (voltage drops)} \tag{3.7}$$

Illustration. In Fig. 3.5 the following values are given:

$R_1 = 2.0$ ohms, $E_1 = 10$ volts
$R_2 = 7.0$ ohms, $E_2 = 24$ volts
$R_3 = 3.5$ ohms, $E_3 = 18$ volts
$R_4 = 7.5$ ohms

Determine the current I and the voltage drop across R_2.

Solution. The batteries having a rise in voltage in the direction of the current arrow will be called positive, while those in the opposite direction will be negative. Thus, E_1 and E_2 are positive, and E_3 will be negative. Hence the total voltage rise is

$$E = E_1 + E_2 - E_3$$
$$= 10 + 24 - 18 = 16 \text{ volts}$$

This voltage is positive and is driving the current I around the series

circuit. If this voltage had come out negative, it would mean that the current I flows in the direction opposite to that of the arrow in Fig. 3.5. This would be indicated by a negative sign before the current.

The total resistance of the circuit is obtained by adding all the resistances of the circuit; thus

$$\begin{aligned} R &= R_1 + R_2 + R_3 + R_4 \\ &= 2 + 7 + 3.5 + 7.5 = 20 \text{ ohms} \end{aligned}$$

Now that the total voltage and resistance of the circuit is known, I, which is the same at all points in a series circuit, is determined by applying Ohm's law; thus

$$I = \frac{E}{R} = \frac{16}{20} = 0.8 \text{ amp} \qquad \textit{Ans.}$$

The voltage across the resistance R_2 can now be found by applying another form of Ohm's law; thus

$$\begin{aligned} IR_2 &= (0.8)(7) = \text{the voltage drop across } R_2 \\ &= 5.6 \text{ volts} \end{aligned} \qquad \textit{Ans.}$$

The voltage drop across any of the other resistances in the circuit can be found in the same manner.

The above example illustrates Kirchhoff's voltage law. In the light of this law, the voltage rises around the circuit can be written equal to the resistance drops around the circuit. Thus by Eq. (3.7),

$$\begin{aligned} E_1 + E_2 - E_3 &= IR_1 + IR_2 + IR_3 + IR_4 \\ &= I(R_1 + R_2 + R_3 + R_4) \end{aligned}$$

Substituting

$$\begin{aligned} 10 + 24 - 18 &= (0.8)(2) + (0.8)(7) + (0.8)(3.5) + (0.8)(7.5) \\ &= 0.8(2 + 7 + 3.5 + 7.5) \end{aligned}$$

Simplifying $16 = 1.6 + 5.6 + 2.8 + 6 = (0.8)(20) = 16$
which proves the truth of the law in one case.

With this law the voltage between any two points in a series circuit can be determined. For instance, the voltage rise from a to b in Fig. 3.5, denoted by E_{ab}, can be written

$$E_{ab} = -IR_4 + E_1 - IR_1 - IR_2$$

Substituting on the right-hand side

$$\begin{aligned} E_{ab} &= -(0.8)(7.5) + 10 - (0.8)(2) - (0.8)(7) \\ &= -6 + 10 - 1.6 - 5.6 \\ &= -3.2 \text{ volts} \end{aligned}$$

This means that the circuit to the left of points a, b can be replaced by -3.2 volts. To check, the voltage in the right-hand part of the circuit from b to a must be the negative to E_{ab}, that is, $+3.2$ volts. The voltage from b to a can be written

$$\begin{aligned} E_{ba} &= E_2 - E_3 - IR_3 \\ &= 24 - 18 - (0.8)(3.5) \\ &= 3.2 \text{ volts} \end{aligned}$$

showing that there is a drop of 3.2 volts from a to b on the left-hand side and a rise of 3.2 volts from b to a on the right-hand side; hence, the voltage rise is equal to the voltage drop around the circuit. Mathematically

$$E_{ab} + E_{ba} = 0$$

Substituting

$$-3.2 + 3.2 = 0$$

This is another way of writing Kirchhoff's voltage law, which in other words says, *the sum of the voltage drops, around a series circuit in a given direction, is zero.* In this form the law is very useful, because of its ease of application. For illustration, in Fig. 3.5 start at point a and go around the circuit in the direction of the current arrow I. There are positive voltage drops and negative voltage rises; thus

$$IR_4 - E_1 + IR_1 + IR_2 - E_2 + E_3 + IR_3 = 0$$

Substituting

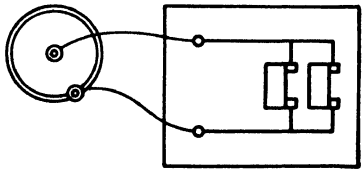
$$(0.8)(7.5) - 10 + (0.8)(2) + (0.8)(87) - 24 + 18 + (0.8)(3.5) = 0$$

$$6 - 10 + 1.6 + 5.6 - 24 + 18 + 2.8 = 0$$

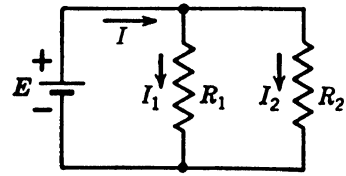
$$34 - 34 = 0$$

Check.

Going around the circuit in the opposite direction, all the signs must be changed, but the result will remain unchanged. This may seem simple here, but if it is not thoroughly understood, difficulties will arise later.



(a) Wiring diagram of a parallel circuit



(b) Schematic diagram of a parallel circuit

FIG. 3.6. Two resistors in parallel with a dry cell.

3.7. Simple Parallel Circuits. When pieces of electrical conductor are connected side by side so that the current divides among them, they are said to be connected in *parallel.* Sometimes a parallel connection is referred to as a *multiple* or *shunt* connection. Figure 3.6 shows two

resistors connected in parallel with a dry cell. A parallel circuit may contain any number of pieces of electrical conductors such as resistors.

3.8. Kirchhoff's Current Law. Parallel circuit problems can also be solved by adding the currents flowing in the various branches to get the total current. In other words, by Kirchhoff's current law, *the sum of the currents flowing up to a point equals the sum of the currents flowing away from that point.* In equation form

$$I = I_1 + I_2 + I_3 + \cdots + I_n \tag{3.8}$$

where I = total current flowing up to the common point, amp
$I_1, I_2, I_3, \ldots I_n$ = the branch currents flowing away from the common point, amp
n = number of branch currents

Example 3.8. In the parallel circuit of Fig. 3.6, let $E = 1.5$ volts, $R_1 = 15$ ohms, and $R_2 = 60$ ohms. Determine the currents flowing in the circuit.

Solution. Since 1.5 volts is across each branch, Ohm's law in Eq. (3.1) can be used to get

$$I_1 = \frac{E}{R_1} = \frac{1.5}{15} = 0.1 \text{ amp} \qquad \textit{Ans.}$$

$$I_2 = \frac{E}{R_2} = \frac{1.5}{60} = 0.025 \text{ amp} \qquad \textit{Ans.}$$

Adding the branch currents, the total current by Eq. (3.8) is

$$I = I_1 + I_2$$
$$= 0.1 + 0.025 = 0.125 \text{ amp} \qquad \textit{Ans.}$$

3.9. Resistances and Conductances in Parallel Circuits. Since the voltage is the same across each branch of a parallel circuit, the current through each branch can be determined by Ohm's law. Substituting the values of voltage and resistance in the current equation (3.8)

$$I = I_1 + I_2 + I_3 + \cdots + I_n$$

results in

$$\frac{E}{R} = \frac{E}{R_1} + \frac{E}{R_2} + \frac{E}{R_3} + \cdots + \frac{E}{R_n}$$

Since the voltage is common to each term, it can be eliminated to get

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_n} \tag{3.9}$$

This equation states that the reciprocal of the equivalent resistance of a parallel circuit is equal to the sum of the reciprocals of the resistances of the individual branches. This equation can also be written

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots + \frac{1}{R_n}} \tag{3.10}$$

where the values are as defined in Eq. (3.9).

For the simple case of two resistors in parallel, the equivalent resistance is equal to the product of the resistance of the individual branches divided by the sum of the resistance of the individual branches. To prove this

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_2 + R_1}{R_1 R_2}$$

or

$$R = \frac{R_1 R_2}{R_1 + R_2} \tag{3.11}$$

where R = equivalent resistance in ohms of two parallel resistors, and R_1 and R_2 are the parallel resistors measured in ohms.

Example 3.9. Determine the equivalent resistance of 600 and 2,400 ohms in parallel.

Solution. By Eq. (3.11)

$$R = \frac{(600)(2{,}400)}{600 + 2{,}400} = \frac{1{,}440{,}000}{3{,}000} = 480 \text{ ohms} \qquad \textit{Ans.}$$

When more than two resistors are connected in parallel, resort can be made to Eq. (3.9) or Eq. (3.10), or the problem can be solved in terms of conductances. Conductance is the reciprocal of resistance and is measured in *mhos*, the word "ohm" spelled backwards. Conductance expresses the ability of a circuit to conduct current; hence a circuit with high resistance has low conductance, and a circuit with low resistance has high conductance. In equation form

$$1 \text{ mho conductance} = \frac{1}{1 \text{ ohm resistance}} \tag{3.12}$$

Using this terminology and letting G stand for the conductance, Eq. (3.9) can be written

$$G = G_1 + G_2 + G_3 + \cdots + G_n \tag{3.13}$$

where $G = \frac{1}{R}$; $G_1 = \frac{1}{R_1}$; $G_2 = \frac{1}{R_2}$; $G_3 = \frac{1}{R_3}$; and $G_n = \frac{1}{R_n}$

If Ohm's law is written in terms of conductances, we have the following three forms:

$$I = GE \tag{3.14}$$

$$G = \frac{I}{E} \tag{3.15}$$

$$E = \frac{I}{G} \tag{3.16}$$

where I = current, amp
G = conductance, mhos
E = voltage, volts

Using conductances in Kirchhoff's current law gives

$$I = I_1 + I_2 + I_3 + \cdots + I_n$$

or

$$GE = G_1E + G_2E + G_3E + \cdots + G_nE \tag{3.17}$$

for the total and respective branch currents.

Example 3.10. Solve Example 3.8 by using conductances.

Solution

$$G_1 = \frac{1}{R_1} = \frac{1}{15} = 0.06667 \text{ mho}$$

$$G_2 = \frac{1}{R_2} = \frac{1}{60} = 0.01667 \text{ mho}$$

$$G = G_1 + G_2 = 0.06667 + 0.01667 = 0.08334$$

By Eq. (3.14)

$$I_1 = EG_1 = 1.5(0.06667) = 0.1 \text{ amp} \quad \textit{Ans.}$$

$$I_2 = EG_2 = 1.5(0.01667) = 0.025 \text{ amp} \quad \textit{Ans.}$$

$$I = EG = 1.5(0.08334) = 0.125 \text{ amp} \quad \textit{Ans.}$$

3.10. Voltages in Parallel. When two or more batteries having the same open-circuit terminal voltage are connected in parallel, the terminal voltage will not be changed, providing like terminals are connected. For a given external load the current drain will be less per battery, and the internal resistance drop for the group will be less. Such connections are commonly used when a large current is needed at a fairly constant voltage.

If the open-circuit terminal voltages are not the same, there will be a circulating current set up in the battery proportional to the difference of the open-circuit terminal voltages. Hence, care should be taken to connect in parallel only batteries with equal open-circuit terminal voltages.

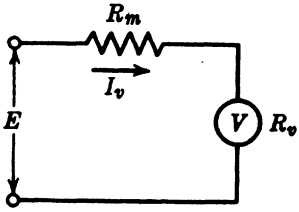
3.11. Electrical Instruments. Voltage Multipliers, Current Shunts, and the Ohmmeter. An *electrical instrument* is a device for measuring the value of an electrical quantity under observation. Instruments such as voltmeters and ammeters are often referred to in industry as "meters." Actually, voltmeters and ammeters are instruments, because they indicate the values at the time of observation.

A voltmeter is an instrument for measuring the voltage between two points in an electric circuit, and an ammeter is an instrument for measuring the current through an electric circuit. On the other hand, watt-hour meters integrate with respect to time the quantity of electricity flowing through a circuit and hence are meters.

Instruments and meters are described on following pages. The purpose of this discussion is to treat the external electric circuits that are used to change the scale readings.

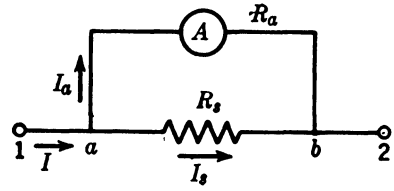
An ideal electrical instrument would not consume power. Therefore, an ideal voltmeter would have *infinite* resistance, and an ideal ammeter would have *zero* resistance. In actual practice, voltmeters are made with very high resistance, while ammeters are made with very low resistance.

Voltmeters are generally made by using resistance in series with a sensitive ammeter that will give full-scale deflection for little current. The resistance that is placed in series with the ammeter is then chosen to



$$R_m = \frac{E}{I_v} - R_v \doteq \frac{E}{I_v}$$

(a)



$$R_s = \frac{I_a}{I - I_a} R_a \doteq \frac{I_a R_a}{I}$$

(b)

(a) Voltmeter with multiplier resistance

(b) Ammeter with shunt resistance

FIG. 3.7. Electrical instrument multipliers and shunts.

permit the full-scale current to flow when the voltage across the instrument is the highest value anticipated. Thus, if we assume that the resistance of the meter itself is negligible, this value of resistance will be given by the maximum voltage to be measured, divided by the current necessary to give full-scale deflection on the meter. If the meter shows full-scale deflection for 1-ma current, it will be necessary to place 1,000 ohms in series with it for each volt to be measured, and such a meter is said to have a sensitivity of "1,000 ohms per volt." Thus, if 100,000 ohms are placed in series with the meter, 100 volts across the instrument will give a full-scale deflection on the meter.

The internal resistance of d-c electrical instruments varies from the order of 100 ohms per volt to the order of 50,000 ohms per volt. In communication work, electrical instruments having a sensitivity of 1,000 ohms per volt are quite common. Such an instrument may be used at one time as a voltmeter by using a series or multiplier resistance, and at another time as an ammeter by using a parallel or shunt resistance. It is therefore desirable to be able to calculate readily the resistance values for multipliers and shunts.

a. Voltage Multipliers. Consider the electrical instrument in Fig. 3.7*a*. The line voltage E is applied to the multiplier resistance R_m in series with the internal meter resistance R_v. By Kirchhoff's voltage law

$$E = I_v R_v + I_v R_m$$

Solving this equation for the multiplier resistance results in the *multiplier-resistance formula*

$$R_m = \frac{E}{I_v} - R_v \tag{3.18}$$

where R_m = series voltage-multiplier resistance, ohms
E = line voltage for full-scale deflection, volts
I_v = full-scale meter current, amp
R_v = internal meter resistance, ohms

For illustration, it is desired to make a 1-ma meter, having a sensitivity of 1,000 ohms per volt, register a full-scale deflection on 500 volts. If the instrument has an internal resistance of 100 ohms, the values can be substituted in Eq. (3.18) to give

$$R_m = \frac{500}{0.001} - 100 = 499{,}900 \text{ ohms} \qquad \textit{Ans.}$$

the resistance value of the external multiplier.

When measuring high voltages, as in the preceding illustration, the external resistance R_m is much larger than the internal resistance R_v. For practical purposes, R_v can be neglected to give

$$R_m \doteq \frac{E}{I_v} \tag{3.19}$$

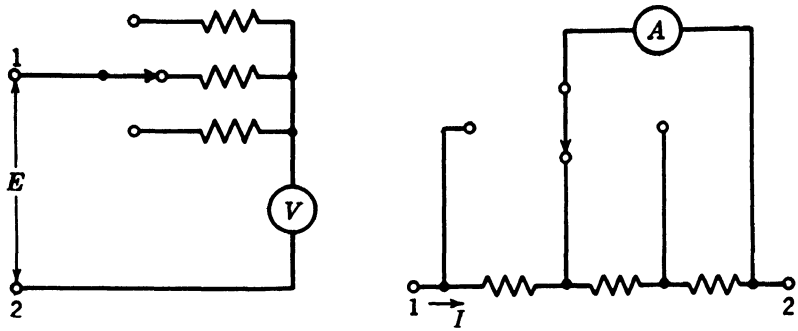
where the values are as defined in Eq. (3.18). If the illustration problem is solved by this approximate formula,

$$R_m \doteq \frac{500}{0.001} = 500{,}000 \text{ ohms} \qquad \textit{Ans.}$$

giving an error of 1 in 4,999 or 0.02 per cent. This approximate formula can also be stated in terms of the meter sensitivity. Since 1,000 ohms is required for each volt, then it will require (1,000)(500) = 500,000 ohms for this voltmeter to give a full-scale deflection of 500 volts.

It is quite common practice to use a series of multiplier resistors with a single low-range voltmeter as shown in Fig. 3.8*a*, thus making it possible by means of a switch to select a number of voltage ranges. In selecting resistors for the multiplier, care must be taken to secure the required accuracy. For a precision job, the resistance should be noninductive (see Chap. 7 for discussion of inductive effects), have a low temperature

coefficient, and be selected to give the exact resistance value by measurement. Commercial resistors may vary as much as ±10 per cent, while precision resistors are readily available with an accuracy of ±1 per cent.



(a) Multirange voltmeter (b) Multirange ammeter

FIG. 3.8. Multirange electrical instruments.

b. Current Shunts. In Fig. 3.7*b*, the instrument A is shunted by the resistance R_s. The line current divides from point a to point b, flowing through the parallel resistors R_a and R_s. Applying Kirchhoff's current law,

$$I = I_a + I_s$$

or in terms of the shunt current

$$I_s = I - I_a \tag{3.20}$$

Furthermore, the voltage drops through the parallel branches can be equated; thus

$$I_s R_s = I_a R_a$$

which in terms of the shunt resistance is written

$$R_s = \frac{I_a}{I_s} R_a \tag{3.21}$$

Substituting the value of I_s from Eq. (3.20) into Eq. (3.21) results in the *shunt-resistance formula*

$$R_s = \frac{I_a}{I - I_a} R_a \tag{3.22}$$

where R_s = shunt current resistance, ohms
I = line current for full-scale deflection, amp
I_a = full-scale meter current, amp
R_a = internal meter resistance, ohms

For illustration, it is desired to make a 1-ma meter, having a sensitivity of 1,000 ohms per volt, register a full-scale deflection of 1 amp. If the

instrument has an internal resistance of 100 ohms, the values can be substituted in Eq. (3.22) to get

$$R_s = \frac{0.001}{1.0 - 0.001} 100 = 0.1001 \text{ ohm} \qquad Ans.$$

the resistance value of the external shunt.

In the measurement of large currents, the resistance of the instrument is usually very high in comparison with the shunt resistance, hence for practical purposes can be neglected. If I_a in the denominator of Eq. (3.22) is neglected,

$$R_s \doteq \frac{I_a}{I} R_a \tag{3.23}$$

where the values are as defined in Eq. (3.22). Solving for the value of the shunt resistance in the preceding illustration by using this approximate formula results in

$$R_s \doteq \frac{0.001}{1.0} 100 = 0.1 \text{ ohm} \qquad Ans.$$

which is in error by 1 part in 1.001 or 0.1 per cent, which in most cases can be neglected.

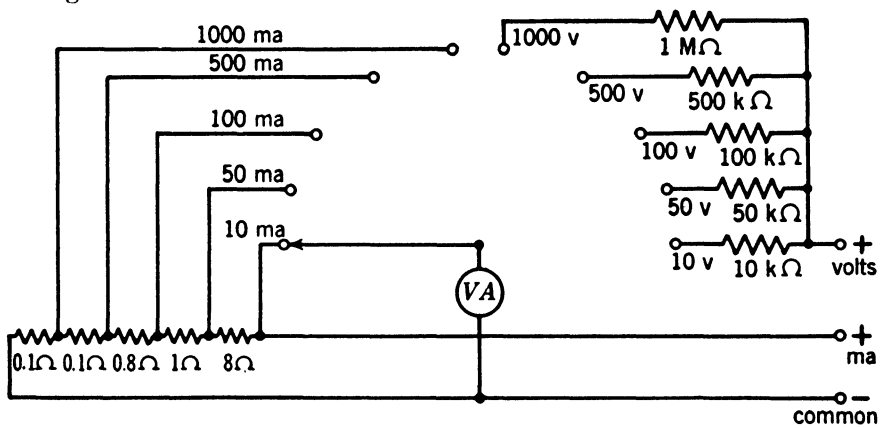


FIG. 3.9. Combination volt-milliammeter circuit. (Values are for the theoretical case of $R_s = 0$ and $R_a = \infty$.)

In many cases the instrument is calibrated with the shunt, and the scale is made to read directly. In the illustration given the instrument should be made to read directly for the value of current flowing in the line.

A number of shunt resistors can be connected by means of a switching arrangement as shown in Fig. 3.8*b* to make a multirange instrument. It will be noted that the shunt resistors are *never* removed from the circuit in this switching scheme. This is an important feature, because if the shunt resistors were in the switching circuit, a poor or open contact might injure

or ruin the meter due to the large current. In the circuit of Fig. 3.8*b*, an open contact will remove the meter from the circuit and the meter indication will drop to zero.

Example 3.11. Using a 0- to 1-ma instrument, design a combination voltmeter and milliammeter circuit. Determine the approximate values of multiplier resistors to make the instrument read a maximum voltage of 10, 50, 100, 500, and 1,000 volts and the approximate values of shunts to make the instrument read 10, 50, 100, 500, and 1,000 ma. The instrument has an internal resistance of 100 ohms and a sensitivity of 1,000 ohms per volt. What are the exact values of resistance required?

Solution. The approximate values computed by using Eqs. (3.19) and (3.23) are shown in the designed circuit of Fig. 3.9. By using Eq. (3.18), the exact values of multiplier resistors are

$$R_m = \frac{10}{0.001} - 100 = 9{,}900 \text{ ohms} \quad \text{(for the 10-volt scale)}$$

$$R_m = \frac{50}{0.001} - 100 = 49{,}900 \text{ ohms} \quad \text{(for the 50-volt scale)}$$

$$R_m = \frac{100}{0.001} - 100 = 99{,}900 \text{ ohms} \quad \text{(for the 100-volt scale)}$$

$$R_m = \frac{500}{0.001} - 100 = 499{,}900 \text{ ohms} \quad \text{(for the 500-volt scale)}$$

$$R_m = \frac{1000}{0.001} - 100 = 999{,}900 \text{ ohms} \quad \text{(for the 1,000-volt scale)}$$

It will be noted that these values are 100 ohms less than the approximate values given in Fig. 3.9. Now the exact values of shunt resistors by Eq. (3.22) are

$$R_s = \frac{0.001}{0.01 - 0.001} 100 = 11.11 \text{ ohms} \quad \text{(for the 10-ma scale)}$$

$$R_s = \frac{0.001}{0.05 - 0.001} 100 = 2.0408 \text{ ohms} \quad \text{(for the 50-ma scale)}$$

$$R_s = \frac{0.001}{0.1 - 0.001} 100 = 1.0101 \text{ ohms} \quad \text{(for the 100-ma scale)}$$

$$R_s = \frac{0.001}{0.5 - 0.001} 100 = 0.2004 \text{ ohm} \quad \text{(for the 500-ma scale)}$$

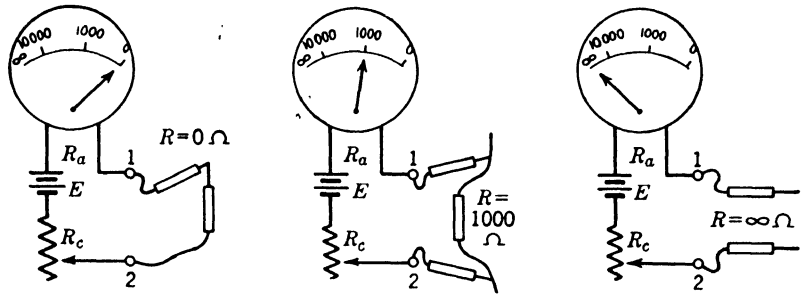
$$R_s = \frac{0.001}{1.0 - 0.001} 100 = 0.1001 \text{ ohm} \quad \text{(for the 1,000-ma scale)}$$

In Fig. 3.9 the values must be added in series to get the total R_s for the respective cases. The error due to the approximate formula increases for the lower current scales.

c. The Ohmmeter. An extremely useful instrument of wide applicability is the direct-reading ohmmeter shown in Fig. 3.10. This instrument can be used to check for open or short circuits and measure resistance values quickly with a fair degree of accuracy. The ohmmeter circuit is often combined with a combination voltmeter-milliammeter circuit to make a very useful portable circuit tester.

The ohmmeter circuit consists of an adjustable zero calibrating resistance R_c and a battery E connected in series with a d-c instrument. The

circuit constants are selected so that the meter can be adjusted to read full scale when the test terminals are shorted ($R = 0$) as shown in Fig. 3.10*a*. If the battery voltage changes, the value of the calibrating resistor R_c has to be readjusted to make the meter read 0 ohms on short circuit. This indicates that the resistance between the two terminals is zero. When the terminals are placed across an unknown resistance, as shown in Fig. 3.10*b*, the meter will read its value directly on the ohm scale. For zero current through the meter, which corresponds to an open circuit, the ohmmeter scale indicates infinity as shown in Fig. 3.10*c*. Therefore, the ohmmeter reads backwards, 0 ohms being at the right-hand end of the scale and ∞ ohms being at the left-hand end of the scale.



(a) With test leads shorted adjust R_c until the meter reads zero

(b) Connect test leads across unknown resistor R and read resistance value on the meter

(c) When test leads are open circuited the resistance is ∞.

FIG. 3.10. Ohmmeter circuit showing calibration and measurement.

The accuracy of the ohmmeter measurements is limited by the accuracy of the instrument and the ability of the operator to read the position of the pointer between the calibrated lines on the scale. The greatest accuracy of the instrument occurs when the unknown resistance is equal to the internal resistance of the ohmmeter circuit, a condition that results in half-scale deflection on the meter. The extreme ends of the scale should be avoided because of their low accuracy. To read low values of resistance, the calibrating resistor R_c should have a low value with a corresponding low value of voltage E, while to read large values of resistance accurately, the value of R_c and E should be made large.

Example 3.12. Design an ohmmeter circuit that will operate with maximum accuracy at 1,000 ohms and use a 0- to 1-ma meter having an internal resistance of 100 ohms.

Solution. In the circuit of Fig. 3.10, the internal resistance must equal an external resistance of 1,000 ohms for maximum accuracy; therefore,

$$R_c + R_a = 1{,}000$$

$$R_c = 1{,}000 - 100 = 900 \text{ ohms} \qquad \textit{Ans.}$$

A rheostat of 1,000 ohms will be satisfactory.

The current flowing in the circuit for maximum accuracy will be 0.5 ma; therefore the battery voltage should be

$$\begin{aligned} E &= 2(R_c + R_a)I \\ &= 2(900 + 100)(0.0005) \\ &= 0.5 \text{ volt} \end{aligned} \qquad \textit{Ans.}$$

Example 3.13. Determine R_c and E in Example 3.12 for a maximum accuracy at 100 kilohms.

Solution

$$R_c = 100{,}000 - R_a = 100{,}000 - 100 = 99{,}900 \qquad \textit{Ans.}$$

$$\begin{aligned} E &= 2(R_c + R_a)I = 2(99{,}900 + 100)(0.0005) \\ &= 50 \text{ volts} \end{aligned} \qquad \textit{Ans.}$$

3.12. Series-parallel Circuits. We are now ready to solve series-parallel circuits, which will involve Ohm's law and Kirchhoff's voltage and current laws.

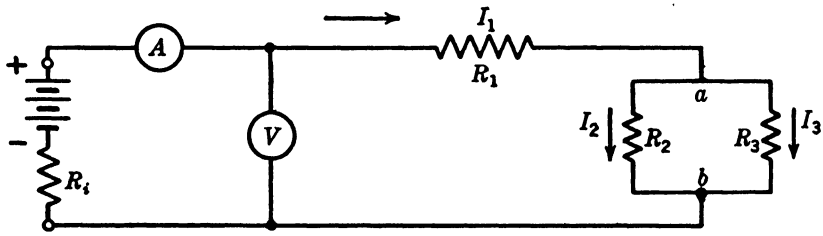


FIG. 3.11. Series-parallel circuit.

Illustration. Let us consider the series-parallel circuit of Fig. 3.11. In this figure we have for illustration the following problem. Given

$$\begin{aligned} E &= 12.3 \text{ volts} = \text{battery voltage} \\ R_i &= 0.2 \text{ ohm}, \qquad R_2 = 12 \text{ ohms} \\ R_1 &= 3.2 \text{ ohms}, \qquad R_3 = 8 \text{ ohms} \end{aligned}$$

Find I_1, I_2, I_3, E_{ab}, and V.

Solution. Since R_2 and R_3 are given, the equivalent resistance is

$$R_{ab} = \frac{R_2R_3}{R_2 + R_3} = \frac{(8)(12)}{8 + 12} = \frac{96}{20} = 4.8 \text{ ohms}$$

This equivalent resistance R_{ab} is in series with R_1 and R_i. Hence, the total resistance is

$$\begin{aligned} R_t &= R_1 + R_i + R_{ab} \\ &= 3.2 + 0.2 + 4.8 = 8.2 \text{ ohms} \end{aligned}$$

By Ohm's law the current flowing in the series circuit is

$$I_1 = \frac{E}{R_t} = \frac{12.3}{8.2} = 1.5 \qquad \textit{Ans.}$$

The terminal voltage can now be found by applying Kirchhoff's voltage law; thus

$$R_i I_1 - E + V = 0$$

or transposing $V = E - R_i I_1$

$$= 12.3 - (0.2)(1.5) = 12.3 - 0.3$$
$$= 12.0 \text{ volts} \qquad \textit{Ans.}$$

The voltage V in the equation was expressed as a voltage drop, since the sign was positive. If the sign of V had been made negative in this equation, the answer would have come out negative, meaning that it is a voltage rise. The voltage V is the terminal voltage of the battery under the particular load condition given. This voltage, of course, varies as the load is varied, owing to the internal resistance of the battery.

We must now find the voltage drop across R_1 in order to arrive at the voltage E_{ab}. This can be expressed by Kirchhoff's voltage law; thus

$$V = I_1 R_1 + E_{ab}$$

or transposing $E_{ab} = V - I_1 R_1 = 12 - (1.5)(3.2)$

$$= 12 - 4.8 = 7.2 \text{ volts} \qquad \textit{Ans.}$$

This could have been obtained without solving for the terminal voltage V; thus

$$E = I_1 R_i + I_1 R_1 + E_{ab}$$

Factoring out I_1

$$E = I_1(R_i + R_1) + E_{ab}$$

Transposing $E_{ab} = E - I_1(R_i + R_1)$

Substituting

$$E_{ab} = 12.3 - 1.5(0.2 + 3.2) = 12.3 - 1.5(3.4)$$
$$= 12.3 - 5 = 7.2 \text{ volts} \qquad \textit{Check.}$$

We are now in a position to determine I_2 and I_3, since we know E_{ab}. By Ohm's law

$$I_2 = \frac{E_{ab}}{R_2} = \frac{7.2}{12} = 0.6 \text{ amp} \qquad \textit{Ans.}$$

and

$$I_3 = \frac{E_{ab}}{R_3} = \frac{7.2}{8} = 0.9 \text{ amp} \qquad \textit{Ans.}$$

3.13. Voltage Dividers or Potentiometers. Voltage potentiometers are devices for varying or adjusting potential. In a simple form, the potentiometer may consist of two resistors connected in series across a battery as shown in Fig. 3.4. The battery voltage E is divided into two parts. One portion of the voltage is across R_1 and the other portion is

across R_2. If a load is placed across R_2, the series-parallel circuit of Fig. 3.12 results. In this potentiometer circuit, R_1 and R_2 can be continuously varied to obtain any desired division of the voltage. The sum

$$R = R_1 + R_2 \tag{3.24}$$

is the total resistance of the potentiometer. Further analysis reveals that

$$E = I_1\left(R_1 + \frac{R_2R_3}{R_2 + R_3}\right) \tag{3.25}$$

and

$$E_3 = I_1\frac{R_2R_3}{R_2 + R_3} \tag{3.26}$$

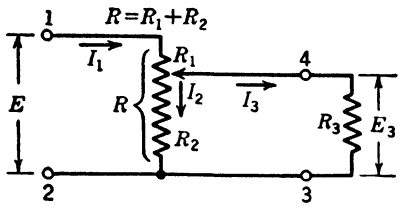


FIG. 3.12. Potentiometer circuit connected to load.

These equations can be expressed in ratio form to eliminate the current I_1; thus

$$\frac{E_3}{E} = \frac{I_1\dfrac{R_2R_3}{R_2 + R_3}}{I_1\left(R_1 + \dfrac{R_2R_3}{R_2 + R_3}\right)} = \frac{R_2R_3}{R_1R_2 + R_2R_3 + R_3R_1} \tag{3.27}$$

Example 3.14. In a radio circuit, a 1-megohm potentiometer is connected across 10 volts. If a 100-kilohm load resistor is connected as shown in Fig. 3.12, where must the potentiometer be set to produce 1 volt across the load?

Solution. By using Eq. (3.27)

$$R_1 + R_2 = 1{,}000{,}000$$

And in Eq. (3.27)

$$\frac{1}{10} = \frac{100{,}000R_2}{R_1R_2 + 100{,}000R_2 + 100{,}000R_1}$$

Upon substituting $R_2 = 1{,}000{,}000/R_1$, this equation can be reduced to the following standard form:

$$R_1^2 + 10R_1 - 9{,}000{,}000 = 0$$

Applying the quadratic formula, the potentiometer must be set so that

$$R_1 = 2{,}995 \text{ ohms} \qquad \textit{Ans.}$$
$$R_2 = 1{,}000{,}000 - 2{,}995 = 997{,}005 \text{ ohms} \qquad \textit{Ans.}$$

3.14. Voltage Bleeders. Voltage dividers or bleeders are often used in conjunction with power supplies to obtain the correct plate voltage for vacuum tubes. This is especially true in radio receivers, where the power dissipated in the bleeder resistance is small. In transmitters it is the usual practice to design the power supply so that bleeders are not necessary.

Let us consider the following practical problem to illustrate how such problems are handled.

Illustration. A d-c power supply of 300 volts is shown in Fig. 3.13. From this power supply it is desired to determine the value of resistances

in a bleeder circuit to supply the required operating voltages for a radio receiver containing three type 58 tubes, three type 56 tubes, and one type 59 output tube. The operating voltages of these tubes are available in charts and books containing tube data. The data in Table 3.1 on these tubes was obtained from such a chart. From this data we can determine the total plate current drain when the plate voltage is 250 volts. The plate current of all the tubes used must be added, giving

$$I_p = (3)(5) + (3)(8.2) + 26 = 65.6 \text{ ma}$$

The screen-grid current drain is

$$I_s = (3)(2) = 6 \text{ ma}$$

The total plate and screen-grid current will return to the bleeder from the cathodes. Since the grids are negative, they will not draw current. We then have the circuit as shown in Fig. 3.13. We still have a parameter to fix arbitrarily, and that is the current I_3. The logical thing to do is to make this current zero, for it is of no practical value other than in stabilizing the circuit voltages if some of the tubes are removed. If $I_3 = 0$, then $R_3 = \infty$, which is an open circuit. We are now ready to determine the values of R_1, R_2, R_4, R_5, and R_6.

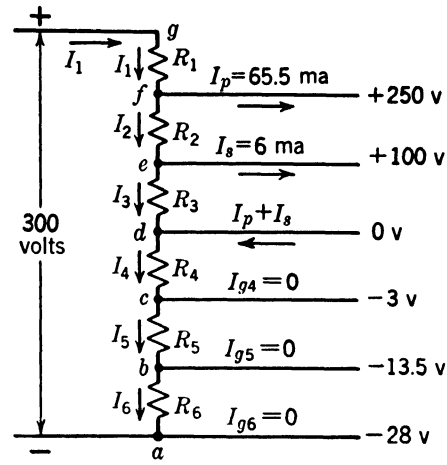


FIG. 3.13. Resistance bleeder.

TABLE 3.1

Tube type	Plate voltage	Screen voltage	Grid voltage	Plate current, ma	Screen current, ma
6P5 GT/G	250	...	−13.5	5.0	
6U7G	250	100	−3	8.2	2.0
59	250	...	−28	26	

Solution. By Kirchhoff's current law, the current at point e is

$$I_2 = I_s + I_3 = 6 + 0 = 6 \text{ ma}$$

Similarly that at point f is

$$I_1 = I_p + I_2 = 65.6 + 6 = 71.6 \text{ ma}$$

Since the grid currents

$$I_{g4} = I_{g5} = I_{g6} = 0$$

we can write by Kirchhoff's current law that the current at d is

$$I_p + I_s = I_4 = 71.6 \text{ ma}$$

and $$I_4 = I_5 = I_6 = 71.6 \text{ ma}$$

Now we can solve for the voltages by applying Kirchhoff's voltage law. The voltage supply between a and g is $E_{ag} = 300$ volts. The total bias between a and d is $E_{ad} = 28$ volts. The plate voltage between d and f is $E_{df} = 250$ volts. Then the voltage E_{fg} across R_1 must make up the difference

$$E_{ag} = E_{ad} + E_{df} + E_{fg}$$

Substituting $$300 = 28 + 250 + E_{fg}$$

$$E_{fg} = 22 \text{ volts}$$

Solving for R_1 by Ohm's law gives

$$R_1 = \frac{E_{fg}}{I_1} = \frac{22}{0.0716} = 307.8 \text{ ohms}$$

The voltage from e to f is the difference of plate and screen voltage or 150 volts. Hence

$$R_2 = \frac{E_{ef}}{I_2} = \frac{150}{0.006} = 25{,}000 \text{ ohms}$$

Similarly

$$R_4 = \frac{E_{ed}}{I_4} = \frac{3.0}{0.0716} = 41.9 \text{ ohms}$$

$$R_5 = \frac{E_{bc}}{I_5} = \frac{10.5}{0.0716} = 146.7 \text{ ohms}$$

$$R_6 = \frac{E_{ab}}{I_6} = \frac{14.5}{0.0716} = 202.5 \text{ ohms}$$

For practical purposes, it is not necessary to make these resistances exact. The following commercial resistance values would be entirely satisfactory:

$R_1 = 300$	(instead of 307.3 ohms)	*Ans.*
$R_2 = 25{,}000$	ohms	*Ans.*
$R_4 = 40$	(instead of 41.9 ohms)	*Ans.*
$R_5 = 150$	(instead of 146.7 ohms)	*Ans.*
$R_6 = 200$	(instead of 202.5 ohms)	*Ans.*

We have yet to determine the safe power rating of these resistances.

$$P_1 = I_1^2 R_1 = (0.0716)^2(307.3)$$

$$= 1.578 \text{ watts} \qquad \text{(use a 2-watt resistor)}$$

$$P_2 = I_2^2 R_2 = (0.002)^2(25{,}000)$$

$$= 1 \text{ watt} \qquad \text{(satisfactory)}$$

$$P_4 = I_4^2R_4 = (0.0716)^2(41.9)$$
$$= 0.2148 \text{ watt} \qquad \text{(use 0.5-watt resistor)}$$
$$P_5 = I_5^2R_5 = (0.0716)^2(146.7)$$
$$= 0.752 \text{ watt} \qquad \text{(use 1-watt resistor)}$$
$$P_6 = I_6^2R_6 = (0.716)^2(202.5)$$
$$= 1.038 \text{ watts} \qquad \text{(use 2-watt resistor)}$$

However, for R_6 a 1-watt resistor would be satisfactory, since the rating would scarcely be exceeded.

Each bleeder problem will be different, and in general the thing to do is to set down everything that is given and then proceed to apply Ohm's and Kirchhoff's laws to determine the unknown elements. If the student thoroughly understands the preceding computation, he should have no more difficulty with this type of problem.

Summary

Closed circuits are normal circuits having a continuous path for current flow.

Open circuits are broken circuits that prevent current flow.

Shorted circuits are circuits in which the current flow is by-passed from its normal path.

Ohm's law states that the current flowing in a circuit is directly proportional to the voltage across the circuit and inversely proportional to the resistance of the circuit. In word equations

$$\text{Amperes} = \frac{\text{volts}}{\text{ohms}}, \qquad \text{Volts} = (\text{amperes})(\text{ohms}), \qquad \text{Resistance} = \frac{\text{volts}}{\text{amperes}}$$

In terms of algebraic symbols,

$$I = \frac{E}{R}, \qquad E = IR, \qquad R = \frac{E}{I}$$

Terminal voltage is the voltage measured at the output terminals of a voltage source. It is equal to the open-circuit voltage minus the internal IR drop.

In a *series circuit*

1. The *total resistance* is the sum of the individual resistances.
2. The *same current* flows in each part of the circuit.
3. Kirchhoff's voltage law states that the sum of the *voltage drops* is equal to the applied voltage. Or, between any two points in a circuit, the voltage drop must be equal regardless of the path.
4. The total applied voltage is the sum of the individual voltage sources.

In a *parallel circuit*

1. The *total conductance* (reciprocal of resistance) is the sum of the parallel branch conductances. The total resistance of a parallel circuit is equal to the reciprocal of the sum of the reciprocals of the individual resistances.
2. The *same voltage* is across each branch.
3. Kirchhoff's current law states that the total current flowing up to a point equals the sum of the *branch currents* flowing away from the point.
4. In a circuit containing similar voltage sources in parallel, the total current in the external circuit is the sum of equal contributions from each similar voltage source.

Electrical instruments indicate present values of the electrical quantity under observation.

Electrical meters integrate and register the electrical quantity with respect to time.

Multirange voltmeters use various large values of series resistance to change the range of the scale.

Multirange ammeters use various small values of parallel resistance to change the range of the scale.

An ohmmeter can be made by connecting a suitable battery and resistance in series with an electrical instrument.

Series-parallel circuits problems can be solved by applying series circuit rules to the series portions and parallel circuit rules to the parallel portions.

Voltage division can be achieved by using a potentiometer or bleeder circuit.

Exercises

3.1. Given a battery with 32.5 volts having an internal resistance of 0.6 ohm. A load is connected that draws 9 amp. Solve for the terminal voltage V and the load resistance R_x (see Fig. 3.3*b*).

3.2. Referring to Fig. 3.6, we have given

$E = 110$ volts
$R_2 = 220$ ohms
$I_1 = 2$ amp

Solve for I, I_2, and R_1.

3.3. In Fig. 3.14, we have given

$R_1 = 1.0$ ohm
$R_2 = 0.5$ ohm
$R_3 = 0.25$ ohm
$R_4 = 0.15$ ohm
$E = 75$ volts

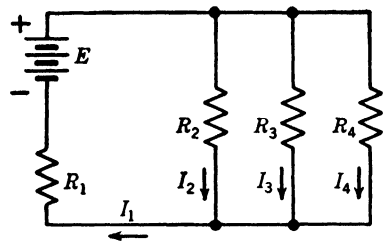


FIG. 3.14. Series-parallel circuit for Exercise 3.3.

Find the conductance G_2, G_3, and G_4 of each parallel circuit and the currents I_1, I_2, I_3, and I_4.

3.4. Given a Weston Model 301, 30-ohm, 0- to 1-ma meter. (*a*) Determine the shunt resistance R_s so that this meter will operate as a 0- to 5-amp ammeter. (*b*) Determine the multiplying series resistance R_m so that this meter will operate as a 0- to 50-volt voltmeter.

3.5. Given the bleeder shown in Fig. 3.15 with

$R_4 = \infty$ ohms
$P = 4.5$ watts
$I_p = 200$ ma
$I_s = 80$ ma

Find R_1, R_2, R_3, and I_1.

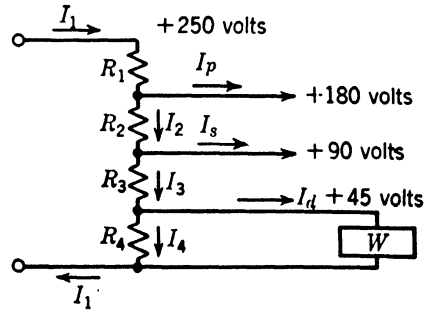


FIG. 3.15. Bleeder for Exercise 5.

CHAPTER 4

DIRECT-CURRENT NETWORKS

Ohm's and Kirchhoff's laws will now be applied to networks that are more complicated than the circuits of the last chapter. The application of equivalent T- and π-sections makes it possible to simplify many rather difficult problems. In addition, the theory and application of the bridge circuit so widely used in making measurements will be given in this chapter.

4.1. Ladder Networks. The ladder network is an extension of the series-parallel circuit treated in the last chapter. The series-parallel circuit in Fig. 3.11 may be considered as a simple two-mesh network. Going a step farther, consider the three-mesh ladder network of Fig. 4.1. An illustrative problem will be solved to show the application of Ohm's and Kirchhoff's laws.

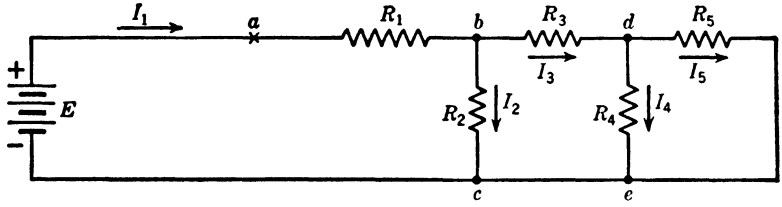


FIG. 4.1. Three-mesh ladder network.

Illustration. With all the resistances and voltages given, it is possible to determine the currents flowing in each branch of the circuit. In Fig. 4.1, we have given

$$R_1 = 60 \text{ ohms}, \qquad R_4 = 300 \text{ ohms}$$
$$R_2 = 600 \text{ ohms}, \qquad R_5 = 150 \text{ ohms}$$
$$R_3 = 300 \text{ ohms}, \qquad E = 15 \text{ volts}$$

Find I_1, I_2, I_3, I_4, and I_5.

Solution. We must solve for the resistance load of the battery E, which is the resistance looking into the circuit between the points a and c.

A general inspection of the circuit indicates that resistances R_4 and R_5 are in parallel. This parallel branch is in series with R_3. Then R_2 is in parallel with this series-parallel branch. Finally, R_1 is in series with this parallel branch. The problem then becomes one of finding, first, the equivalent resistance of a parallel circuit and then adding it to a series resistance, and so on.

The resistance R_{45} of the parallel circuit containing R_4 and R_5 by Eq. (3.11) of the last chapter is

$$R_{45} = \frac{R_4 R_5}{R_4 + R_5} = \frac{(150)(300)}{300 + 150} = \frac{45{,}000}{450} = 100 \text{ ohms}$$

Adding the series resistance R_3 gives

$$R_{be} = R_3 + R_{45} = 300 + 100 = 400 \text{ ohms}$$

The parallel circuit between b and c has a resistance R_{bc}, which is

$$R_{bc} = \frac{R_{be} R_2}{R_2 + R_{be}} = \frac{(400)(600)}{600 + 400} = \frac{240{,}000}{1{,}000} = 240 \text{ ohms}$$

The sum of this resistance R_{bc} and R_1 gives the equivalent resistance R_{ac}, the load on the battery.

$$R_{ac} = R_{bc} + R_1 = 240 + 60 = 300 \text{ ohms}$$

The current I_1 by Ohm's law is

$$I_1 = \frac{E}{R_{ac}} = \frac{15}{300} = 0.05 \text{ amp} = 50 \text{ ma} \qquad Ans.$$

Note. If volts and ohms are used in the equation of Ohm's law, the current will be in amperes. The preceding equation could have been written using millivolts and ohms. Then the answer would be in milliamperes as follows:

$$I_1 = \frac{15{,}000}{300} = 50 \text{ ma}$$

The student may find this form more convenient for radio engineering problems where small currents are in continuous use. In any event it is essential that the student keep clearly in mind the system of units being used.

The voltage drop from b to c by Kirchhoff's voltage law gives

$$E = I_1 R_1 + R_{bc}$$

Substituting and solving for E_{bc} gives

$$E_{bc} = 15 - (0.05)(60) = 15 - 3.0 = 12 \text{ volts}$$

By Ohm's law

$$I_2 = \frac{E_{bc}}{R_2} = \frac{12}{600} = 0.020 \text{ amp.} = 20 \text{ ma} \qquad Ans.$$

Using Kirchhoff's current law at the point b results in

$$I_1 = I_2 + I_3$$

Substituting and solving for I_3

$$I_3 = I_1 - I_2 = 50 - 20 = 30 \text{ ma} \qquad \textit{Ans.}$$

The voltage drop E_{de} is found by applying Kirchhoff's voltage law to the circuit *cbde* as follows:

$$E_{cb} = I_3R_3 + E_{de}$$

Substituting and solving

$$\begin{aligned} E_{de} &= E_{cb} - I_3R_3 = 12 - (0.03)(300) \\ &= 12 - 9 = 3 \text{ volts} \end{aligned}$$

Note that E_{cb} is a voltage rise, while E_{bc} as expressed above is a voltage drop. In other words, going from b to c, one experiences a drop in voltage, while going from c to b, a rise in voltage is experienced. Applying Ohm's law, we find the current

$$I_4 = \frac{E_{ed}}{R_4} = \frac{3}{300} = 0.01 \text{ amp} = 10 \text{ ma} \qquad \textit{Ans.}$$

The current I_5 can be obtained by applying Kirchhoff's current law.

$$I_5 = I_3 - I_4 = 30 - 10 = 20 \text{ ma} \qquad \textit{Ans.}$$

Or, applying Ohm's law,

$$I_5 = \frac{E_{ed}}{R_5} = \frac{3}{150} = 0.02 \text{ amp} = 20 \text{ ma} \qquad \textit{Check.}$$

To summarize the answer:

$$\begin{aligned} I_1 &= 50 \text{ ma} & I_4 &= 10 \text{ ma} \\ I_2 &= 20 \text{ ma} & I_5 &= 20 \text{ ma} \\ I_3 &= 30 \text{ ma} \end{aligned}$$

This type of problem is not difficult to work, but the work must be accurate. If more than three meshes were involved, it would be handled in the same manner, by first solving for the equivalent resistance of the circuit starting at the far end from the battery and then going back through the circuit solving for the various currents.

The student may inquire as to the practical value of this type of problem. Ladder-type network attenuators are a good example of the practical application.

Illustration. In Fig. 4.2 it is desired to design a resistance network for obtaining multiples of 1 mv, such as 10 mv, 100 mv, and 1,000 mv or 1 volt.

If we consider this a purely voltage device, which means that there is no current drain across the output terminals, the values of resistance

in the network can be any practical value. However, the output load will take a little current, so it is well to make the resistance network of rather low values of resistance; that is, when looking into the network from the output terminals, the internal resistance should have a low value. The open-circuit voltage will be correct in any case, but when there is an appreciable load applied, the drop in the internal resistance becomes appreciable. If this resistance is made small, the drop will be small, and

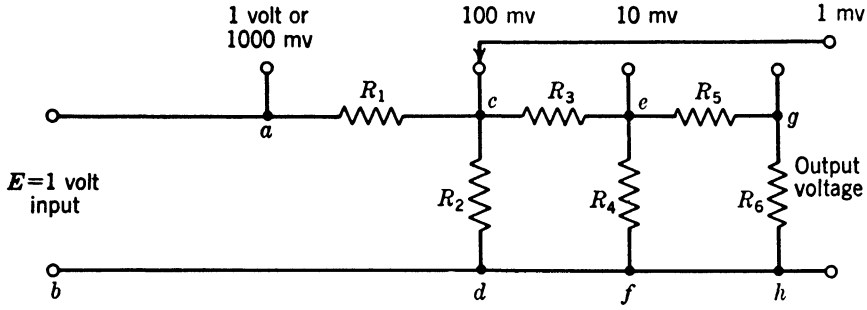


FIG. 4.2. Ladder-type attentuator.

hence the error in using the open-circuit terminal voltage will be less. The limiting factor in making this resistance small is the power consumption from the input voltage. If very small resistances are used, large currents will be necessary to produce the required IR drops to give the correct voltages.

With these facts in mind, we are ready to design the network. Arbitrarily we select $R_6 = 9$ ohms, because it is low in comparison with output loads normally used and will not consume a prohibitive amount of current. The current I_6 by Ohm's law is

$$I_6 = \frac{\text{millivolts}}{\text{ohms}} = \frac{1}{9} = 0.111 \text{ ma}$$

This current flows through R_5, so

$$R_5 = \frac{10 - 1}{0.111} = 81 \text{ ohms}$$

It would be desirable to select the value of R_4 so that the network will have uniform shunt resistors. We have the resistance R_4 to select arbitrarily. If we keep the above limiting conditions in mind, 10 ohms should be a suitable value for this resistance; thus,

$$I_4 = {}^{10}\!/_{10} = 1.0 \text{ ma}$$

Then the current through R_3 is

$$I_3 = I_4 + I_5 = 1.0 + 0.111 = 1.111 \text{ ma}$$

The resistance R_3 becomes

$$R_3 = \frac{100 - 10}{1.111} = 81 \text{ ohms}$$

Similarly, we will make $R_2 = 10$ ohms and get

$$I_2 = {}^{100}\!/_{10} = 10 \text{ ma}$$

Solving for I_1 gives

$$I_1 = I_2 + I_3 = 10 + 1.111 = 11.111 \text{ ma}$$

R_1 can now be found and is

$$R_1 = \frac{1{,}000 - 100}{11.111} = 81 \text{ ohms}$$

This completes the solution of this voltage-multiplying ladder-type network. This type of network has valuable applications in a-f and r-f attenuators.

Later on in the course, attenuators will be discussed in more detail, considering decibel attenuation and impedance-matching problems involved in their design.

4.2. Equivalent T- and π-sections. All linear network problems can be handled by the proper application of Ohm's and Kirchhoff's laws; however, in complicated networks this procedure becomes very tedious. By applying the theory of equivalent T- and π-sections, most complicated networks can be quickly reduced to a simple network. To make this possible, the equations for transforming a π- to a T-section and vice versa will be developed.

THEOREM I. A three-element T-section can be replaced by an equivalent π-section and vice versa if the relationships given in Fig. 4.3 *are maintained.*

To prove this theorem, consider the T- and π-sections of Fig. 4.3. The resistance between the various terminals, as designated by the subscripts, can be written for the T- and π-sections as follows:

$$R_{13} = R_1 + R_3 = \frac{R_a(R_b + R_c)}{R_a + R_b + R_c} \tag{4.1}$$

$$R_{12} = R_1 + R_2 = \frac{R_b(R_a + R_c)}{R_a + R_b + R_c} \tag{4.2}$$

$$R_{23} = R_2 + R_3 = \frac{R_c(R_a + R_b)}{R_a + R_b + R_c} \tag{4.3}$$

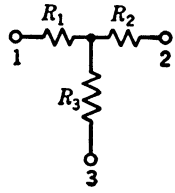
Here we have three equations in three unknowns, R_1, R_2, and R_3 for the T-section or R_a, R_b, and R_c for the π-section. By simultaneous equations it is possible to find a solution.

Let us first solve for the values of resistance in the T-section in terms of the π-section resistances. By adding Eqs. (4.1) and (4.2)

$$2R_1 + R_2 + R_3 = \frac{2R_aR_b + R_bR_c + R_bR_c}{R_a + R_b + R_c}$$

Now subtracting Eq. (4.3) removes R_2 and R_3; thus

$$2R_1 = \frac{2R_aR_b + \cancel{R_aR_c} + \cancel{R_bR_c} - \cancel{R_aR_c} - \cancel{R_bR_c}}{R_a + R_b + R_c}$$

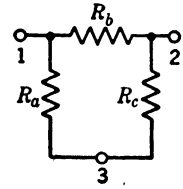


$$R_1 = \frac{R_aR_b}{R_a + R_b + R_c}$$

$$R_2 = \frac{R_bR_c}{R_a + R_b + R_c}$$

$$R_3 = \frac{R_cR_a}{R_a + R_b + R_c}$$

(a) T-section with values in terms of an equivalent π-section



$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$

(b) π-section with values in terms of an equivalent T-section

FIG. 4.3. Equivalent T- and π-sections.

Canceling terms and dividing by 2 gives

$$R_1 = \frac{R_aR_b}{R_a + R_b + R_c} \tag{4.4}$$

By adding Eqs. (4.2) and (4.3), subtracting Eq. (4.1), canceling terms, and dividing by 2

$$R_2 = \frac{R_bR_c}{R_a + R_b + R_c} \tag{4.5}$$

By adding Eqs. (4.1) and (4.3), subtracting Eq. (4.2) and simplifying,

$$R_3 = \frac{R_cR_a}{R_a + R_b + R_c} \tag{4.6}$$

These are the required equations for transforming any π-section to its equivalent T-section.

The values of resistance in the π-section in terms of the T-section resistances can be obtained from Eqs. (4.4), (4.5), and (4.6) by using the following procedure: Multiplying Eqs. (4.4) and (4.5) together gives

$$R_1R_2 = \frac{(R_aR_bR_c)(R_b)}{(R_a + R_b + R_c)^2}$$

Multiplying Eqs. (4.5) and (4.6)

$$R_2R_3 = \frac{(R_aR_bR_c)(R_c)}{(R_a + R_b + R_c)^2}$$

Multiplying Eqs. (4.4) and (4.6)

$$R_3R_1 = \frac{R_aR_bR_c(R_a)}{(R_a + R_b + R_c)^2}$$

Adding these three equations

$$R_1R_2 + R_2R_3 + R_3R_1 = \frac{(R_aR_bR_c)(R_a + R_b + R_c)}{(R_a + R_b + R_c)^2}$$

$$= \frac{R_aR_bR_c}{R_a + R_b + R_c} \tag{4.7}$$

Dividing Eq. (7.4) by Eq. (4.5) and turning the equation around results in

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2} \tag{4.8}$$

Dividing Eq. (4.7) by Eq. (4.6) and turning the equation around gives

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} \tag{4.9}$$

Dividing Eq. (4.7) by Eq. (4.4) and transposing yields

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} \tag{4.10}$$

Example 4.1. In a π-section as shown in Fig. 4.3*b*, let $R_a = 300$, $R_b = 500$, and $R_c = 200$. What are the values in the equivalent T-section?

Solution. In the equivalent T-section, by Eq. (4.4),

$$R_1 = \frac{R_aR_c}{R_a + R_b + R_c} = \frac{(300)(500)}{300 + 500 + 200} = 150 \text{ ohms} \qquad \textit{Ans.}$$

By Eq. (4.5),

$$R_2 = \frac{R_bR_c}{R_a + R_b + R_c} = \frac{(500)(200)}{300 + 500 + 200} = 100 \text{ ohms} \qquad \textit{Ans.}$$

And finally by Eq. (4.6),

$$R_3 = \frac{R_cR_a}{R_a + R_b + R_c} = \frac{(200)(300)}{300 + 500 + 200} = 60 \text{ ohms} \qquad \textit{Ans.}$$

Example 4.2. Show that a T-section with $R_1 = 150$ ohms, $R_2 = 100$ ohms, and $R_3 = 60$ ohms can be converted to an equivalent π-section.

Solution. Substituting in Eq. (4.8)

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

$$= \frac{(150)(100) + (100)(60) + (60)(150)}{100} = 300 \text{ ohms} \qquad \textit{Ans.}$$

Since the numerator $R_1R_2 + R_2R_3 + R_3R_1$ is the same in all three equations, we have by Eq. (4.9)

$$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} = \frac{30{,}000}{60} = 500 \text{ ohms} \qquad \textit{Ans.}$$

Similarly, by Eq. (4.10),

$$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} = \frac{30{,}000}{150} = 200 \text{ ohms} \qquad \textit{Ans.}$$

4.3. Simplifying Complicated Networks.

THEOREM II. Any complicated resistance network having two input and two output terminals can be reduced to an equivalent T- or π-section.

a. By Step Procedure. By using Theorem I to change T-sections to equivalent π-sections and vice versa, it is possible to reduce a complicated

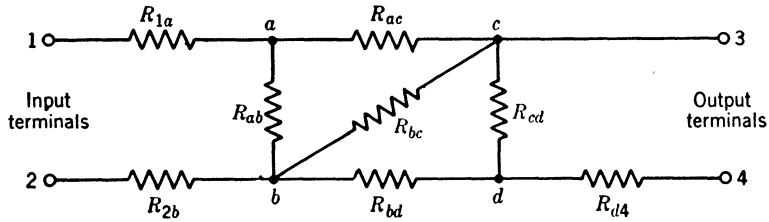


FIG. 4.4. Complicated four-terminal network.

network to a simple T- or π-section. In addition to this type of transformation, resistances in series must be added and resistances in parallel must be added at various steps in the problem.

Illustration. Reduce the four-terminal complicated network of Fig. 4.4 to an equivalent T- and π-section when the values are as follows:

$R_{1a} = 13$ ohms	$R_{bc} = 20$ ohms
$R_{2b} = 8$ ohms	$R_{bd} = 50$ ohms
$R_{ab} = 24$ ohms	$R_{cd} = 30$ ohms
$R_{ac} = 50$ ohms	$R_{d4} = 36$ ohms

Solution. Step 1. Replace the π-section between points *b*, *c*, and *d* with its equivalent T-section as shown in Fig. 4.5. By the equations in Fig. 4.3 for changing a π-section to its equivalent T-section, we have

$$R_{ce} = \frac{R_{bc}R_{cd}}{R_{bc} + R_{bd} + R_{cd}}$$

Substituting and solving,

$$R_{ce} = \frac{(20)(30)}{20 + 50 + 30} = \frac{600}{100} = 6 \text{ ohms}$$

Similarly, $$R_{de} = \frac{R_{bd}R_{cd}}{R_{bc} + R_{bd} + R_{cd}} = \frac{(50)(30)}{100} = 15 \text{ ohms}$$

and $$R_{be} = \frac{R_{bc}R_{bd}}{R_{bc} + R_{bd} + R_{cd}} = \frac{(20)(50)}{100} = 10 \text{ ohms}$$

Step 2. Since the same current flows in at terminal 1 as flows out at terminal 2, the same current flows through R_{1a} and R_{2b}. Since these resistances are in series, they can be added, giving

$$R_{1a}' = R_{1a} + R_{2b} = 13 + 8 = 21 \text{ ohms}$$

An inspection of Fig. 4.5 (Step 1) indicates that the same current flows in R_{bc} as flows in R_{ac}; hence R_{ac}' of Step 2 is the sum

$$R_{ac}' = R_{bc} + R_{ac} = 10 + 50 = 60 \text{ ohms}$$

Also, for the same reason, R_{cd} and R_{d4} can be moved to the other side of

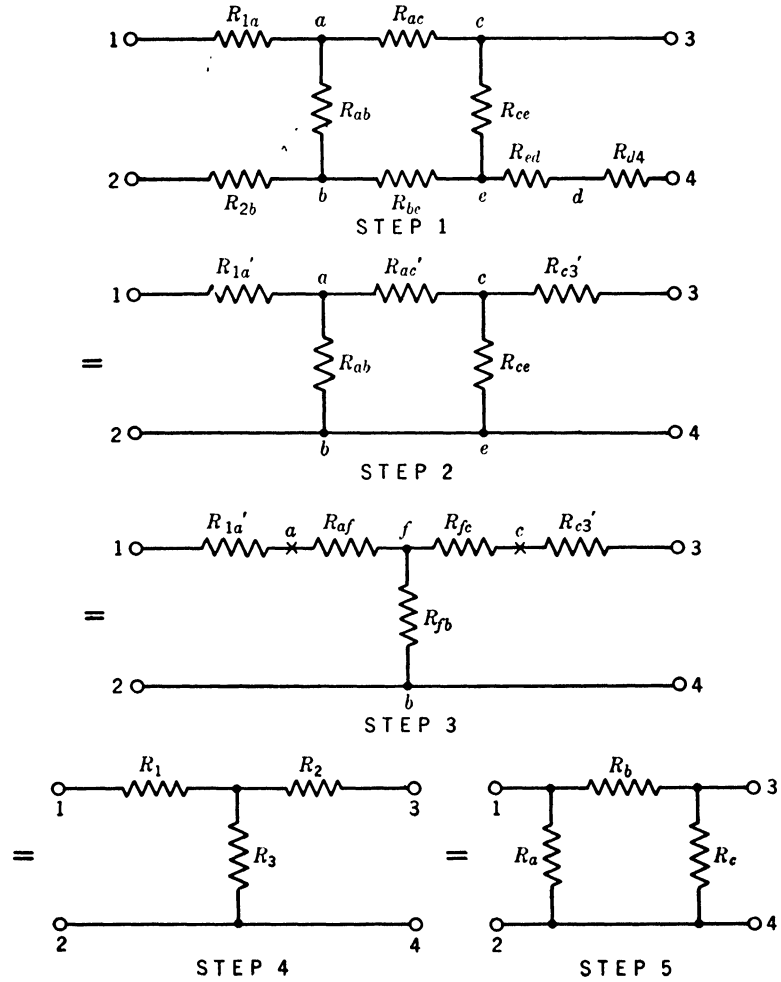


FIG. 4.5. Steps in reducing the complicated network of Fig. 4.4 to its equivalent T- and π-section.

the circuit, giving R_{c3}' shown in Step 2.

$$R_{c3}' = R_{ed} + R_{d4} = 15 + 36 = 51 \text{ ohms}$$

Step 3. The π-section between a, c, and b of Step 2 can now be replaced by the T-section of Step 3. Thus

$$R_{af} = \frac{R_{ac}'R_{ab}}{R_{ab} + R_{ac}' + R_{ce}}$$

Substituting and solving,

$$R_{af} = \frac{(60)(24)}{24 + 60 + 6} = \frac{1{,}440}{90} = 16 \text{ ohms}$$

Similarly $$R_{fc} = \frac{R_{ac}'R_{ce}}{R_{ab} + R_{ac}' + R_{ce}} = \frac{(60)(6)}{90} = 4 \text{ ohms}$$

and $$R_{fb} = \frac{R_{ab}R_{ce}}{R_{ab} + R_{ac}' + R_{ce}} = \frac{(24)(6)}{90} = 1.6 \text{ ohms}$$

Step 4. The resistance R_1 of Step 4 is obtained by adding R_{1a}' and R_{af} of Step 3. Thus,

$$R_1 = R_{1a}' + R_{af} = 21 + 16 = 37 \text{ ohms} \qquad \textit{Ans.}$$

Similarly

$$R_2 = R_{fc} + R_{c3}' = 4 + 51 = 55 \text{ ohms} \qquad \textit{Ans.}$$

Then $$R_3 = R_{fb} = 1.6 \text{ ohms} \qquad \textit{Ans.}$$

This is the equivalent T-section that was desired.

Step 5. The T-section of Step 4 can be changed into the π-section of Step 5 by applying the conversion equations as follows:

$$R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$

Substituting and solving,

$$R_a = \frac{(37)(55) + (55)(1.6) + (1.6)(37)}{55}$$

$$= \frac{2{,}035 + 88 + 59.2}{55} = \frac{2{,}182.2}{55}$$

$$= 39.68 \text{ ohms} \qquad \textit{Ans.}$$

Similarly $$R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3} = \frac{2{,}182.2}{1.6}$$

$$= 1{,}364 \text{ ohms} \qquad \textit{Ans.}$$

and $$R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1} = \frac{2{,}182.2}{37}$$

$$= 58.98 \text{ ohms} \qquad \textit{Ans.}$$

This completes the reduction of the complicated network to the equivalent T- and π-sections.

In actual practice, it is very much more convenient to solve this type of problem from open- and short-circuit measurements when the measuring equipment is available. This type of problem will now be treated.

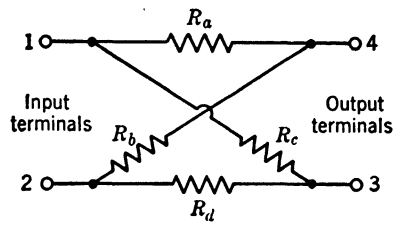
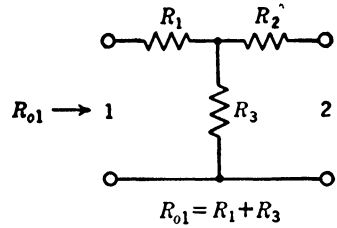
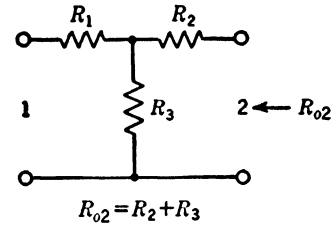


FIG. 4.6. Lattice resistance network.

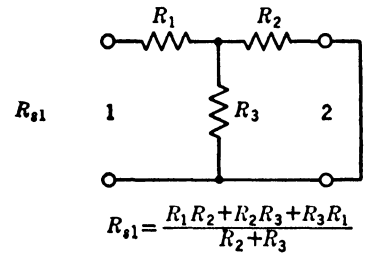
b. By Resistance Measurements. Consider the complicated network of Fig. 4.6. How can it be reduced to an equivalent T- or π-section? Since no T- or π-section can be found, Theorem I can not be applied. Theorem II, however, states that a solution is possible. To find the answer, let us determine if it is possible to



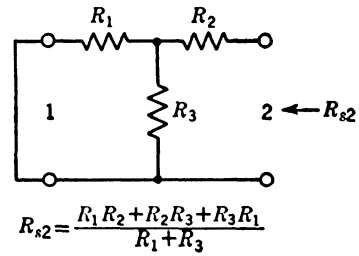
(a) Open-circuit measurements at input terminals



(b) Open-circuit measurements at output terminals



(c) Short-circuit measurements at input terminals



(d) Short-circuit measurements at output terminals

FIG. 4.7. Open- and short-circuit resistance measurements to determine equivalent T-section for an unknown four-terminal complicated network.

make measurements on the network that can be converted into equivalent T- and π-sections.

If the input and output terminals are available, it is possible to make open- and short-circuit measurements at the respective pairs of terminals as shown in Fig. 4.7. This results in four equations as follows:

$$R_{01} = R_1 + R_3 \tag{4.11}$$

$$R_{02} = R_2 + R_3 \tag{4.12}$$

$$R_{s1} = R_1 + \frac{R_2R_3}{R_2 + R_3} = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2 + R_3} \tag{4.13}$$

$$R_{s2} = R_2 + \frac{R_1R_3}{R_1 + R_3} = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1 + R_3} \tag{4.14}$$

With these equations it is possible to solve for R_1, R_2, and R_3, the resistances of the equivalent T-section. To do this, multiply Eqs. (4.11) and (4.14) together and cancel the $R_1 + R_3$ quantity, leaving

$$R_{01}R_{s2} = R_1R_2 + R_2R_3 + R_3R_1 \tag{4.15}$$

Similarly, multiplying Eqs. (4.12) and (4.13) results in

$$R_{02}R_{s1} = R_1R_2 + R_2R_3 + R_3R_1 \tag{4.16}$$

Rewriting Eqs. (4.11) and (4.12)

$$R_1 = R_{01} - R_3 \tag{4.17}$$

$$R_2 = R_{02} - R_3 \tag{4.18}$$

Now substituting these values in Eq. (4.16)

$$R_{02}R_{s1} = (R_{01} - R_3)(R_{02} - R_3) + (R_{02} - R_3)R_3 + R_3(R_{01} - R_3)$$

Expanding and canceling terms

$$R_{02}R_{s1} = R_{01}R_{02} - R_3^2$$

Solving for R_3

$$R_3^2 = R_{01}R_{02} - R_{02}R_{s1}$$

Hence

$$R_3 = \sqrt{R_{02}(R_{01} - R_{s1})} \tag{4.19}$$

Substituting Eq. (4.15) in Eqs. (4.17) and (4.18)

$$R_1 = R_{01} - \sqrt{R_{02}(R_{01} - R_{s1})} \tag{4.20}$$

$$R_2 = R_{02} - \sqrt{R_{02}(R_{01} - R_{s1})} \tag{4.21}$$

These last three equations are the desired equations for determining the resistance of the equivalent T-section.

It is sometimes desirable to obtain the equivalent π-section as shown in Fig. 4.8c. Referring to Eq. (4.16), it is seen that $R_{02}R_{s1}$ can replace the numerator of Eq. (4.8), and by Eq. (4.21), $R_{02} - \sqrt{R_{02}(R_{01} - R_{s1})}$ can be used to replace the denominator; thus

$$R_a = \frac{R_{02}R_{s1}}{R_{02} - \sqrt{R_{02}(R_{01} - R_{s1})}} \tag{4.22}$$

Similarly, Eqs. (4.8) and (4.19) substituted in Eq. (4.9) gives

$$R_b = \frac{R_{02}R_{s1}}{\sqrt{R_{02}(R_{01} - R_{s1})}} \tag{4.23}$$

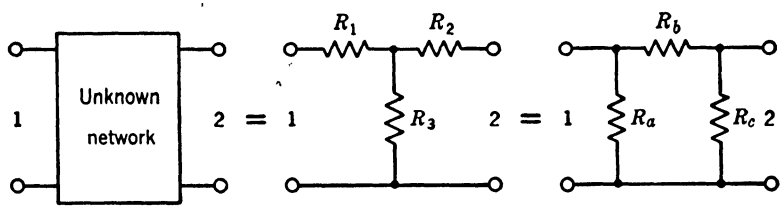
Finally, Eqs. (4.8) and (4.20) substituted in Eq. (4.10) give

$$R_c = \frac{R_{02}R_{s1}}{R_{01} - \sqrt{R_{02}(R_{01} - R_{s1})}} \tag{4.24}$$

It is of interest to note that only three measurements were required to determine the equivalent T-section or π-section. If R_{s2} had been measured instead of R_{s1}, the value of R_{s2} could be determined by equating Eqs. (4.15) and (4.16) to get

$$R_{01}R_{s2} = R_{02}R_{s1} \tag{4.25}$$

It is therefore possible to determine the equivalent T-section or π-section by using R_{01}, R_{02}, and R_{s2}.



(a) Four-terminal complex network having input terminals 1 and output terminals 2.

(b) Equivalent T-section.

$$R_1 = R_{01} - R_3$$

$$R_2 = R_{02} - R_3$$

$$R_3 = \sqrt{R_{02}(R_{01} - R_{s1})}$$

(c) Equivalent π-section.

$$R_a = \frac{R_{02}R_{s1}}{R_2}$$

$$R_b = \frac{R_{02}R_{s1}}{R_3}$$

$$R_c = \frac{R_{02}R_{s1}}{R_1}$$

FIG. 4.8. Networks and associated equations for equivalent T- and π-sections of a complicated four-terminal network.

Example 4.3. Develop an equation for R_3 when the R_{s1} resistance measurement is absent.

Solution. From Eq. (4.19)

$$R_3 = \sqrt{R_{02}(R_{01} - R_{s1})} = \sqrt{R_{01}R_{02} - R_{02}R_{s1}}$$

Substituting for R_{s1} from Eq. (4.25)

$$R_3 = \sqrt{R_{01}R_{02} - R_{02}\left(\frac{R_{01}R_{s2}}{R_{02}}\right)}$$

$$= \sqrt{R_{01}(R_{02} - R_{s2})} \qquad \textit{Ans.}$$

Example 4.4. Develop an equation for R_b that does not contain R_{s1}.

Solution. Substituting the value of R_3 from Example 4.3 in the denominator of Eq. (4.23), and substituting the value of R_{s1} from Eq. (4.25) in the numerator results in

$$R_b = \frac{R_{02}\left(\frac{R_{01}R_{s2}}{R_{02}}\right)}{\sqrt{R_{01}(R_{02} - R_{s2})}} = \frac{R_{01}R_{s2}}{\sqrt{R_{01}(R_{02} - R_{s2})}} \qquad \textit{Ans.}$$

4.4. Simultaneous Equations Applied to a Two-mesh Circuit. It is perhaps well at this point to show how simultaneous equations may be applied to a two-mesh circuit. This can be effectively done by working a problem.

Illustration. Two batteries with different internal resistances are connected to a load as shown in Fig. 4.9, with

$R_1 = 20$ ohms $\quad E_1 = 20$ volts
$R_2 = 10$ ohms $\quad E_2 = 30$ volts
$R_3 = 20$ ohms

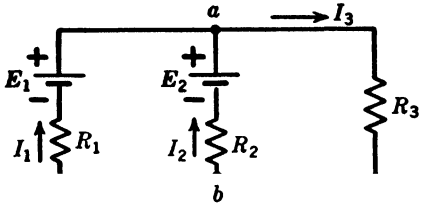


FIG. 4.9. Two-mesh circuits.

Find the currents I_1, I_2, and I_3 flowing in the three circuits by mesh-circuit theory.

Solution. First, applying Kirchhoff's voltage law to the outside circuit, then to the right-hand mesh gives two simultaneous equations. The third is obtained by applying Kirchhoff's current law at point *a*.

$$E_1 = I_3R_3 + I_1R_1 \tag{4.26}$$
$$E_2 = I_3R_3 + I_2R_2 \tag{4.27}$$
$$I_1 + I_2 = I_3 \tag{4.28}$$

Perhaps the easiest way to solve such a set of equations is to substitute in all the known values first and then eliminate all but one unknown. Solving the problem by this method, let us substitute in Eqs. (4.26) and (4.27) as follows:

$$20 = 20I_3 + 20I_1 \tag{4.29}$$
$$30 = 20I_3 + 10I_2 \tag{4.30}$$

If Eq. (4.30) were multiplied by 2, the coefficient of I_1 of Eq. (4.29) would be the same as the coefficient of I_2 in Eq. (4.30). Adding the two equations would give a term $20I_1 + 20I_2$, but from Eq. (4.12), this would be equal to $20I_3$, and we would have eliminated all but I_3. Performing this operation gives

	$20 = 20I_3 + 20I_1$	(4.29)
2 times Eq. (4.30)	$60 = 40I_3 + 20I_2$	(4.31)
Adding	$80 = 60I_3 + 20I_1 + 20I_2$	(4.32)
20 times Eq. (4.28)	$20I_3 = 20I_1 + 20I_2$	
Subtracting	$80 - 20I_3 = 60I_3$	
Or	$80I_3 = 80$	
	$I_3 = 1$ amp	*Ans.*

This value of current can be substituted in Eq (4.29) or Eq. (4.30) to obtain one of the other currents. Substituting I_3 in Eq. (4.30)

$$30 = 20 + 10I_2$$
$$I_2 = 1 \text{ amp} \qquad \textit{Ans.}$$

Substituting these values of current in Eq. (4.28) gives

$$I_1 = I_3 - I_2 = 1 - 1 = 0 \text{ amp} \qquad \textit{Ans.}$$

This shows that battery E_1 is supplying no energy to the load. If the voltage of E_2 were larger, current would actually flow backward through battery E_1. The solution of the above equations would result in a negative current I_1, meaning that the current flows in the opposite direction to the arrow for I_1 in Fig. (4.9). If the battery voltage E_1 were larger than 20 volts, it would help supply energy to the load. The same would be true if the voltage of E_2 were smaller.

4.5. Edison Circuit. The above type of circuit is of value in many electrical engineering problems. For instance, the Edison three-wire circuit as shown in Fig. 4.10 is essentially this type of a circuit. When the loads R_1 and R_2 are equal in value and the voltages E_1 and E_2 are equal, no current will flow through the ammeter A in the neutral or center circuit. This type of circuit is common for supplying power to consumers. For example, if E_1 and E_2 are 110-volt lines, and if R_1 and R_2 represent the load of two consumers, the current in the grounded lead I_n will be zero when each consumer has the same power load. This in effect permits the power company to supply the load at 220 volts and hence at a lower line current. Consequently the line resistance losses are less, because the current is just half what it would be in a single 110-volt line. The line power loss in this case is reduced to one-fourth of what it would be if both customers were supplied by a single 110-volt line. This system also furnishes 220 volts to consumers desiring the higher voltage.

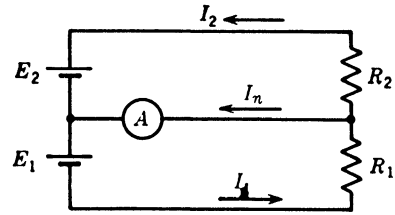


FIG. 4.10. Edison circuit.

4.6. Bridge Circuits. Direct-current resistance bridge circuits are widely used in measuring instruments and as integral parts of electrical networks. The Wheatstone bridge circuit of Fig. 4.11 and the lattice attenuating network of Fig. 4.6 illustrate typical applications of this principle.

A *Wheatstone bridge* will measure with precision resistance values from a fraction of an ohm to many thousands of ohms. To understand how this is accomplished, let us study the theory underlying the Wheatstone bridge. Referring to the bridge circuit of Fig. 4.11, it will be noted that it is an

from the starting point a, R_3 and R_4 are along the right-hand branch and also in the numerator and denominator of the right-hand side of Eq. (4.38).

A common practice in commercial bridge circuits is to make the ratio R_1/R_2 equal to unity, a decimal fraction, or a multiple of 10. When this ratio is unity, R_3/R_4 must also be unity; hence, if R_4 is the unknown, it is equal to the known value of R_3. If $R_1/R_2 = 0.1$, then $R_3/R_4 = 0.1$, and the values of the unknown R_4 are ten times the value of R_3. However, if $R_1/R_2 = 10$, then $R_3/R_4 = 10$; hence the unknown R_4 is 0.1 times the value of R_3. In this type of bridge, the arms R_1 and R_2 are called the *ratio arms*, while the arm R_3 is known as the *rheostat arm*. The ratio-arm dial is calibrated so that the unknown resistance value is the product of the ratio-arm and rheostat-arm readings; thus,

$$R_4 = R_3\left(\frac{R_2}{R_1}\right) = (\text{rheostat arm})(\text{ratio arm}) \tag{4.39}$$

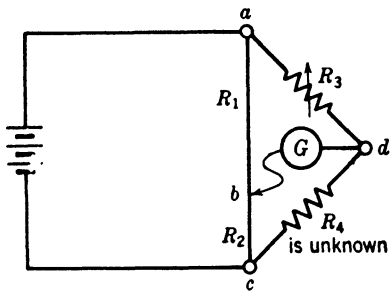
A commercial bridge of this type is illustrated in Fig. 4.12.

Example 4.5. In the measurement of an unknown resistance, the ratio arm reads 0.001 and the rheostat arm reads 4,739. What is the value of the unknown resistance?

Solution. By Eq. (4.39),

$$R_4 = (4{,}739)(0.001) = 4.739 \text{ ohms} \qquad \textit{Ans.}$$

A common laboratory type of Wheatstone bridge, known as the *slidewire* Wheatstone bridge, is shown in Fig. 4.13. In this bridge, the resistances R_1 and R_2 consist of a single piece of resistance wire having a uniform cross section. The point b is established by a sliding contact, which can be moved along the wire from a to c until a balance is obtained. Since the resistance of a uniform wire is proportional to its length, the equation can be written



$$R_4 = \frac{R_2}{R_1} R_3$$

FIG. 4.13. Slide-wire Wheatstone bridge circuit.

$$R_4 = R_3\frac{R_2}{R_1} = R_3\left(\frac{\text{length } bc}{\text{length } ab}\right) \tag{4.40}$$

If scales are marked off along the slide-wire, the lengths ab and bc can be read directly.

Example 4.6. A uniform resistance wire 1 meter long is used as the ratio arm of a slide-wire bridge. If R_3 in Fig. 4.13 is 100 ohms, and if a balance is obtained when the slider contact is 39 cm from a, what is the value of the unknown resistance?

Solution. By Eq. (4.40),

$$R_4 = 100\,\frac{61}{39} = 156.41 \text{ ohms} \qquad \textit{Ans.}$$

4.7. Murray-loop Measurements. The bridge-circuit principle can be used to determine the distance to a ground or short circuit on a communication line. One method is to use the Murray-loop measuring circuit of Fig. 4.14. If l is the length of the wire in miles, and r_1 and r_2 are the resistances per mile of wires 1 and 2, respectively, then the resistance

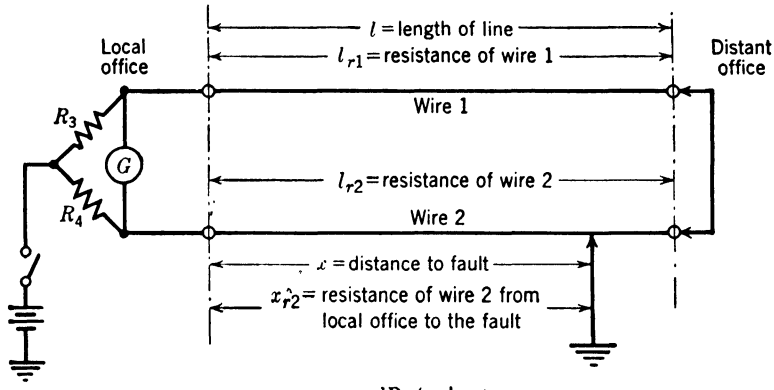


FIG. 4.14. Murray-loop measuring circuit.

of the clear wire is lr_1, and $(l - x)r_2$ is the resistance from the fault to the distant office. xr_2 is the resistance from the fault to the local office. Substituting these values in the fundamental Wheatstone bridge equation,

$$\frac{R_3}{R_4} = \frac{lr_1 + (l - x)r_2}{xr_2}$$

Solving for x

$$x = \frac{lR_4(r_1 + r_2)}{r_2(R_3 + R_4)} \tag{4.41}$$

where x = distance from local office to fault, miles
l = distance between local and distant office, miles
r_1 = resistance per mile of clear wire, ohms
r_2 = resistance per mile of faulty wire, ohms
R_3 = resistance of bridge arm, ohms
R_4 = resistance of bridge arm, ohms

For the special case of wires 1 and 2 having the same resistance per mile, let $r = r_1 = r_2$. Then Eq. (4.41) reduces to

$$x = \frac{2lR_4}{R_3 + R_4} \tag{4.42}$$

The distances in Eqs. (4.41) and (4.42) can be expressed in feet if it is more convenient.

Example 4.7. One wire of a two-wire telephone line is grounded between two offices 20,000 ft apart. If a Murray-loop measurement reveals that $R_3 = 4{,}896$ and $R_4 = 1{,}000$, how far is the fault from the local office?

Solution. By Eq. (4.43)

$$x = \frac{(2)(20{,}000)(1{,}000)}{1{,}000 + 4{,}896} = 6{,}784 \text{ ft}$$

which is the distance from local office to the fault. *Ans.*

4.8. Varley-loop Measurements. The Varley loop is another bridge circuit used to determine the distance to a fault on a communication line. This circuit is more flexible than the Murray loop in that it can be used to measure the total resistance of the loop or be used as a Wheatstone bridge

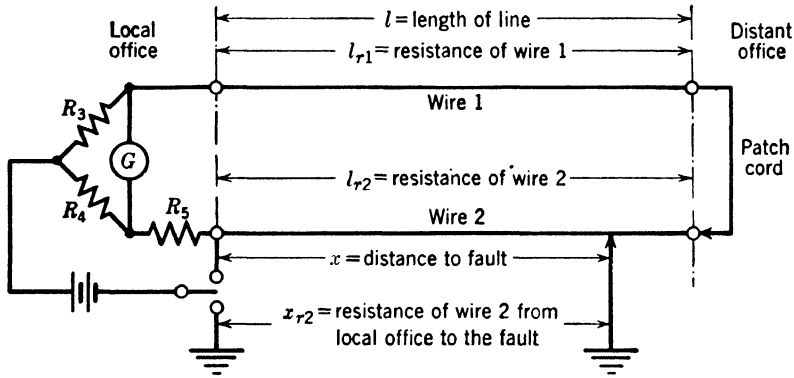


FIG. 4.15. Varley-loop measuring circuit.

to measure any resistance. Referring to Fig. 4.15, it will be noted that the resistance R_5 has been added, along with a different switching arrangement. With the switch in the down position for the Varley-loop condition of operation, the resistance R_5 must be added to xr_2 in the fundamental Wheatstone bridge equation to get

$$\frac{R_3}{R_4} = \frac{lr_1 + (l - x)r_2}{R_5 + xr_2}$$

Solving for x

$$x = \frac{lR_4(r_1 + r_2) - R_3R_5}{r_2(R_3 + R_4)} \tag{4.43}$$

where the terms are defined in Eq. (4.41), except R_5, which is the added resistance in the bridge arm as shown in Fig. 4.15.

Example 4.8. Write the Varley-loop equation for the special condition of both the clear and faulty wire having the same resistance.

Solution. Let $r = r_1 = r_2$ in Eq. (4.43); then

$$x = \frac{2lR_4r - R_3R_5}{r(R_3 + R_4)} \quad \textit{Ans.}$$

Example 4.9. Write the equation for the unknown when the Varley-loop bridge is used as an ordinary Wheatstone bridge.

Solution. Let R_x be the unknown, then by the fundamental Wheatstone bridge equation

$$R_x = \frac{R_3 R_5}{R_4} \qquad Ans.$$

4.9. The Potentiometer. Another extremely useful application of the bridge circuit is the *potentiometer*, which can be used to measure an unknown voltage or current to an accuracy of 1 part in 1 million, or better. A schematic diagram of a potentiometer is shown in Fig. 4.16. A potentiometer is used in conjunction with a standard cell, which is a form of battery that maintains its voltage very constant for many years or under widely varying temperatures. The calibrated slide-wire of the instrument is very carefully designed, and calibrated to 1 part in 1 million. To use the instrument, the standard cell is first connected to the terminals, and the slider set at the point A, which is the point where the calibration of the slide-wire exactly equals the voltage of the standard cell, which is supplied by the manufacturer. The variable resistance R_1 is then adjusted to vary the current from the supply battery until the galvanometer reads zero. At this point the instrument is correctly adjusted, and the voltage at any point on the slide-wire is very accurately known.

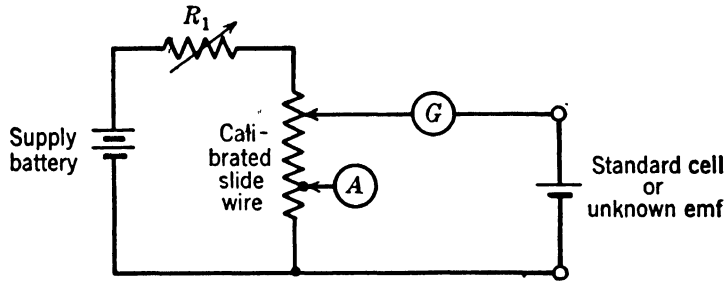


FIG. 4.16. Potentiometer schematic.

To measure an unknown voltage, the standard cell is disconnected, and the unknown substituted in its place. When the slider is then adjusted so that the galvanometer again reads zero, the unknown voltage equals the slider setting exactly. Not only is it possible to measure unknown voltages very exactly in this manner, but the potentiometer has the additional advantage that it is equivalent to an infinite-resistance voltmeter. Since no current flows from the unknown voltage source when the slider is correctly adjusted, the effect is the same as though it were being measured by a voltmeter of infinite resistance.

Summary

A *ladder network* consists of a number of series-parallel circuits, which makes the final network have the appearance of a ladder.

A *T-section* can be replaced by an equivalent π-*section* and vice versa by using the equations of Fig. 4.3.

Any *complex resistance* network with two input and two output terminals can be replaced by an equivalent T- or π-section.

Resistance values can be moved from one input terminal to the other input terminal, since the same current flows through it. Resistances can be moved, similarly, from one output terminal to the other output terminal.

A *lattice resistance* network can be converted to an equivalent T- or π-section by using open- and short-circuit measurements.

Any four-terminal resistance network can be converted to an equivalent T- or π-section by using open- and short-circuit measurements.

Simultaneous equations can be used to solve for the currents of a network if the voltages and resistance values are known.

An *Edison circuit* permits the distribution of power more efficiently with three wires than could be done with two wires. This result is due to doubling the voltage and halving the current being transmitted.

The *Wheatstone bridge* is the basic type from which other bridges are derived.

The fundamental Wheatstone bridge equation is $R_1/R_2 = R_3/R_4$ for a circuit as shown in Fig. 4.11.

The slide-wire takes the place of R_1 and R_2 in the slide-wire bridge.

The *Murray* and *Varley loops* are circuit modifications of the Wheatstone bridge.

Exercises

4.1. In Fig. 4.1 an ammeter is inserted in series with resistance R_5, and a reading of 1.5 amp is noted. The resistances of the network are given as follows:

$R_1 = 30$ ohms $\quad R_4 = 150$ ohms
$R_2 = 300$ ohms $\quad R_5 = 75$ ohms
$R_3 = 150$ ohms

Find the voltages E_{ca} and E_{cb}.

4.2. If the output terminals of the network in Fig. 4.2 are connected to a load of 20 ohms and the switch turned to the 1-mv terminal, what will be the actual voltage output?

4.3. Given the T-section of Fig. 4.3*a* with $R_1 = 80$ ohms, $R_2 = 120$ ohms, $R_3 = 100$ ohms, find the equivalent π-section.

4.4. Given the π-section of Fig. 4.3*b* with $R_a = 1{,}600$ ohms, $R_b = 800$ ohms, $R_c = 2{,}000$ ohms, find the equivalent T-section.

4.5. Simplify the complex network of Fig. 4.4 by first reducing the π-section between terminals *abc* to an equivalent T-section.

4.6. In Fig. 4.6 if $R_a = 100$ ohms, $R_b = 200$ ohms, $R_c = 200$ ohms, and $R_d = 100$ ohms, determine the equivalent T-section.

4.7. In Fig. 4.9 the following values are given:

$R_1 = 20$ ohms $\quad E_1 = 10$ volts
$R_2 = 10$ ohms $\quad E_2 = 30$ volts
$R_3 = 20$ ohms

Determine the currents I_1, I_2, and I_3.

4.8. In Fig. 4.11, if $R_1 = 640$ ohms, $R_2 = 360$ ohms, and $R_3 + R_4 = 1{,}500$ ohms, what is the value of R_3 and R_4 that will give a zero potential difference between points b and d?

4.9. In Fig. 4.13, if the resistance wire ac is 100 cm long and the slider is 87 cm from end a for a balance, what is the resistance of the unknown if $R_3 = 1{,}000$ ohms?

4.10. In the Murray-loop measuring circuit of Fig. 4.14, the length of the line is not known; however, the loop resistance measures R_l ohms. Derive the equation for the distance to the fault.

CHAPTER 5

MAGNETIC CIRCUITS AND FIELDS

It has been known for hundreds of years that pieces of iron and steel, under certain conditions, would exert attractive forces on other bits of iron and steel in their vicinity. This force was first observed in bits of loadstone, a natural magnetic material found in Magnesia, a province of Thessaly in Greece. The name "magnetism" is derived from the name of this locality. Magnetism was first studied scientifically by the English physicist Gilbert. Oersted made the fundamental observation of the connection between magnetism and an electric current, and Michael Faraday discovered how a moving magnet can be made to produce an electric current.

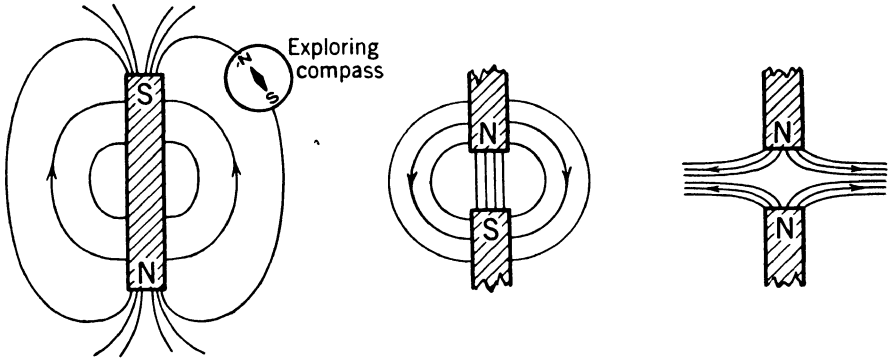
In this chapter we study the relationships between the forces produced by magnets on each other, the production of magnetic fields by electric currents, and the production of electric currents by moving magnetic fields. The various units and terms employed in the study of magnetism are introduced and defined.

5.1. Permanent Magnets. All magnets can be classified as either permanent or temporary. When we speak of a "magnet," we generally mean a *permanent* magnet or, in other words, a magnet that will maintain its magnetizing force in the absence of any outside source of energy. However, if a piece of soft iron, or annealed steel, is touched to a permanent magnet, it will itself attract pieces of iron for as long as the contact is maintained, or as long as it is held very close to the permanent magnet. When the latter is removed, however, the soft iron loses practically all its magnetism. Therefore it is called a *temporary* magnet.

5.2. Magnetic Field about Permanent Magnets. The ability of a permanent magnet, such as the horseshoe or bar magnet, to pick up bits of iron, is well known. If a bar magnet is placed under a piece of glass covered with iron filings, the filings will arrange themselves in lines as shown in Fig. 5.1*a*. The iron filings line up along the *magnetic lines of force* making up the magnetic field around the magnet. These lines of force are said to flow from the north pole to the south pole in the air and back to the north pole through the magnet. The north pole of an exploring compass, if placed in the magnetic field of a permanent magnet, will point toward the south pole as shown in Fig. 5.1*a*. If a compass needle is suspended freely and away from any magnetic field, its north pole always points

toward the earth's geographic North Pole. Now, since the north pole of the compass needle always seeks the south pole of another magnet the earth's geographic North Pole may be considered to be the south pole of a large magnet. That is, the earth itself behaves like a magnet whose north and south poles are the geographic South and North Poles of the earth respectively. The earth's magnetic poles are near, but are not located exactly at, the geographic poles.

5.3. Magnetic Attraction and Repulsion. If the north pole of one magnet is placed near the south pole of another magnet, as shown in Fig. 5.1*b*, the lines of force flow from the north pole to the south pole. These



(a) Bar magnet (b) Opposite poles (c) Like poles

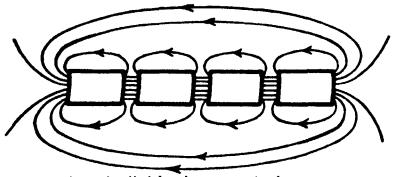
FIG. 5.1. Magnetic field about permanent magnets.

lines of force tend to shorten themselves, like rubber bands, with the result that unlike magnetic poles attract each other. Conversely, if like poles are placed near each other, as shown in Fig. 5.1*c*, the lines of force must find their way to south poles, and in so doing they squeeze sideways against each other, with the result that like magnetic poles repel each other.

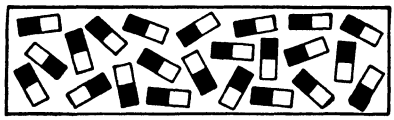
5.4. Theory of Magnetism. In a magnetic circuit, the magnetic flux is said to flow around the magnetic circuit, as an electric current flows around an electric circuit. However, in the magnetic circuit, it appears that there is only a distortion in space, while in an electric circuit, material particles called "electrons" are in actual movement around the circuit. Although magnetic flux does not represent the flow of anything material, such as electrons, it can in many cases be treated like electric current. They are both vector quantities having magnitude and direction.

A very important property of magnetic lines of force is that they are *continuous* about a complete magnetic circuit. This fact can be demonstrated by sawing a bar magnet into a number of pieces, as shown in Fig. 5.2*a*. No matter how many pieces are made from the original magnet, each individual piece becomes a perfect magnet. It will also be noted

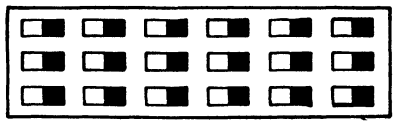
that every line of force has no end. If this idea is carried further, a magnet can be thought of as being made up of many small individual magnets. If a piece of iron has no distinct magnetic poles, it is said to be *unmagnetized.* In this case the small individual magnets can be thought of as being arranged in a random fashion as shown in Fig. 5.2*b*. By placing this piece of iron in a strong magnetic field, it is possible to line up the individual magnets to make the piece of iron magnetized as shown in Fig. 5.2*c*. The piece of iron reaches *magnetic saturation* when increasing the magnetic field further does not increase the magnetic properties of the iron.



(*a*) Individual magnets in a broken bar magnet



(*b*) Unmagnetized iron



(*c*) Magnetized iron

FIG. 5.2. Composition of permanent magnets.

5.5. Magnetic Intensity. In discussing magnetic fields it is necessary to state both the strength and the direction of the field. The strength of a magnetic field is generally given in terms of the effect it would have on a unit north pole. A *unit magnetic pole is* defined as a *pole* that *if placed at a distance of* 1 *cm in vacuum from an exactly similar pole would repel it with a force of* 1 *dyne.* Naturally, it is never possible to separate a north pole from its corresponding south pole, but the concept of a unit pole free to move by itself is very useful in describing magnetic fields. If a unit pole were placed anywhere in a magnetic field, it would move along the lines of force from the north poles to the south poles of the magnets that produce the magnetic field.

The strength of the magnetic field or the magnetic field intensity at any point in free space is defined as *the force in dynes that the magnetic field would exert on a unit pole placed at that point.* This strength is generally stated in terms of the number of lines of flux per sq cm at right angles to the magnetic field at any point. The number of lines of flux per sq cm at any point in the magnetic field is defined as *the number of dynes that would be exerted on a unit north pole at that point.* The unit of magnetic intensity in the cgs system of units is the *oersted.* The flux lines are by convention continuous; that is, each line of flux arising in a magnet and drawn from the north pole to the south pole of the magnet is assumed to continue from the south pole to the north pole within the magnet. Thus, there are as many flux lines within the magnet as there are outside the magnet.

5.6. Magnetic Flux. The amount of flux flowing around a magnetic circuit is measured by the number of lines of force threading through the circuit. This is similar to measuring the current in an electric circuit by determining the number of amperes flowing in the circuit. A *maxwell* is defined as *one line of force.* One line of force is not a discrete thing but merely a unit of measure. In other words, a magnetic circuit may have only a fraction of a line of force flowing around the circuit. Lines of force are also useful in describing the shape of the magnetic field as well as its intensity. Flux is a vector quantity since it has both magnitude and direction.

5.7. Flux Density. In electric circuits the current density is found by dividing the current by the cross-sectional area of the conductor. Similarly in magnetic circuits the flux density is found by dividing the flux by the cross-sectional area through which the flux flows. The flux density or magnetic induction is expressed in lines, or maxwells per sq cm or gausses in the cgs system. In equation form

$$B = \frac{\Phi}{A} \tag{5.1}$$

where B = flux density, gausses
Φ = total flux lines passing through the area, maxwells
A = cross-sectional area normal to the direction of the flux, sq cm.

5.8. Magnetomotive Force. In an electric circuit, emf forces current around the electric circuit, while in the magnetic circuit, mmf forces flux through the magnetic circuit. It has been found experimentally that the flux established by a coil carrying current and linking a magnetic circuit is directly proportional to the current and number of turns in the coil. This statement assumes that the coil turns are all in the same direction for if some turns are wound in the opposite direction, their effect must be subtracted. For illustration, if one-half the turns are wound in the opposite direction, no mmf will be produced. A coil wound in this fashion has no perceptible magnetic field and is said to be noninductively wound. Many resistance coils are wound in this manner.

In the electric circuit emf is measured in volts and represented by E, while in the magnetic circuit mmf is measured in gilberts and represented by $\mathfrak{F}$. The mmf produced by one abampere flowing around one turn is 4π gilberts. In equation form

$$\mathfrak{F} = 4\pi NI \tag{5.2}$$

where $\mathfrak{F}$ = magnetomotive force (mmf), gilberts
N = number of turns of wire on the coil
I = current in coil of wire, abamperes

If I is expressed in amperes this equation becomes $\mathfrak{F} = 0.4\pi NI$. The term 4π enters this equation owing to the geometry in the derivation.

The area of a sphere is 4π times the square of the radius. Starting at point a in Fig. 5.3 and moving to point b counterclockwise around the conductor, the mmf increases by πI gilberts. Following the circular flux line through points c and d and again returning to point a, the mmf becomes $4\pi I$ for one turn, that is, once around the conductor.

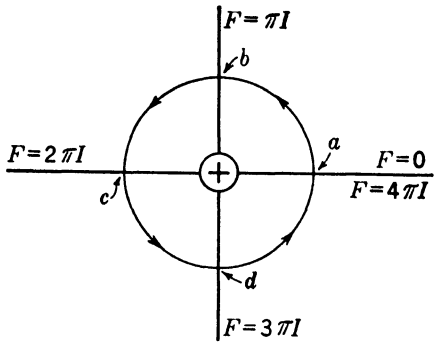


FIG. 5.3. Map showing circular flux line and equipotential radial magnetomotive force lines.

If a small circular flux line is followed close to the conductor in Fig. 5.3 the result obtained will be the same as if a large circular flux line is followed far away from the conductor. This shows that the number of gilberts per unit length may vary widely. It is convenient, when speaking of magnetic circuits, to use the term *field intensity*, which is the magnetic potential drop per unit length and is measured in gilberts per centimeter length or oersted. In equation form

$$H = \frac{\mathfrak{F}}{l} \tag{5.3}$$

where H = field intensity or magnetic force, oersteds
$\mathfrak{F}$ = magnetomotive force, gilberts
l = length of magnetic circuit, cm

5.9. Permeability. A given magnetic field intensity will produce several thousand more flux lines in soft iron than in the same area in air. The ratio of the flux density to field intensity is called *permeability*. *Permeability* is *a measure of the ease with which magnetic lines of force can be established in a substance.* In equation form:

$$\mu = \frac{B}{H} \tag{5.4}$$

where μ = permeability of the material
= 1 for air
B = flux density in the magnetic material, gausses, or lines/sq cm
H = field intensity, oersteds
= lines/sq cm in air

Permeability of all nonmagnetic materials is 1. The permeability of magnetic materials varies widely with the material and with the value of field intensity. Cast iron, nickel, and cobalt have a permeability of several hundred; that of steel is several thousand; that of soft iron

is of the order of 10,000, while that of permalloy may reach the order of 100,000. When sheet steel is slightly magnetized, the permeability may be of the order of only a few hundred, increase to several thousand when moderately magnetized, and as the magnetization is increased, the permeability may drop off rapidly.

Example 5.1. A total flux of 5000 maxwells exists in a permalloy core of 0.25 sq cm cross-sectional area. If the field intensity is 2 oersteds, determine the flux density and permeability.

Solution

By Eq. (5.1)

$$B = \frac{\Phi}{A} = \frac{5{,}000}{0.25} = 20{,}000 \text{ gausses} \qquad \textit{Ans.}$$

By Eq. (5.4)

$$\mu = \frac{B}{H} = \frac{20{,}000}{2} = 10{,}000 \qquad \textit{Ans.}$$

5.10. Reluctance. In an electric circuit, resistance opposes the flow of current, and in a like manner reluctance in a magnetic circuit opposes the establishment of a magnetic field. The reluctance of a magnetic path is directly proportional to the length of the path and inversely proportional to the area. Reluctance also depends upon the permeability of the material used in the magnetic circuit. In equation form, the reluctance of a magnetic path in the cgs system can be written

$$\mathcal{R} = \frac{l}{\mu A} \tag{5.5}$$

where $\mathcal{R}$ = reluctance, rel,* or gilberts/maxwell.

The term reluctance is usually used qualitatively rather than quantitatively because the permeability has such wide variations that the solutions of problems are usually made on a graphical basis.

Example 5.2. A toroid core of cast iron has a cross-sectional area of 0.8 sq in and a mean length of 25 in. If the permeability is 300, determine the reluctance of the toroid.

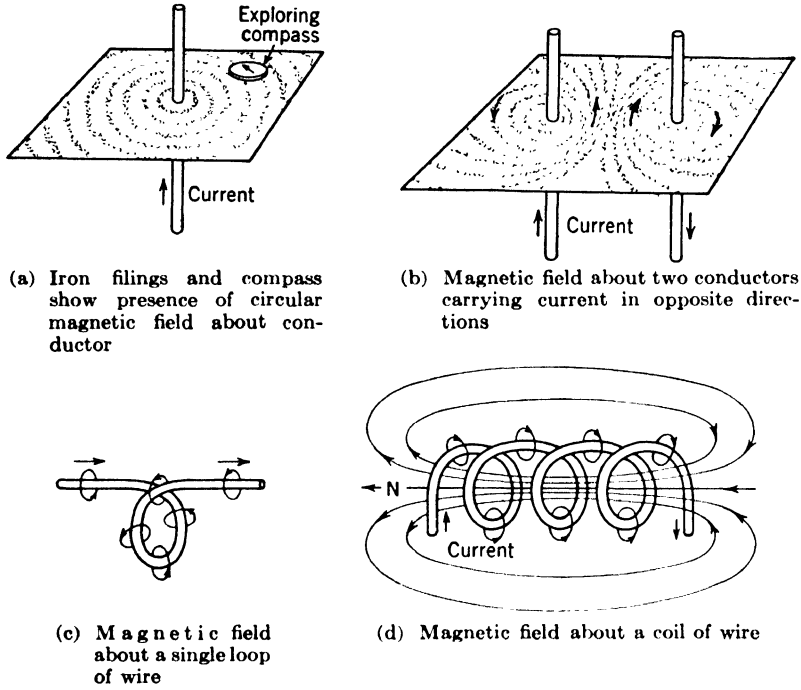
Solution. By Eq. (5.5)

$$\mathcal{R} = \frac{l}{\mu A} = \frac{(25)(2.54)}{(300)(0.8)(6.45)} = 0.0427 \text{ rel} \qquad \textit{Ans.}$$

5.11. Magnetic Field Around Conductors. If iron filings are sprinkled on a horizontal cardboard through which a vertical conductor carrying current is passed, the iron filings will arrange themselves in concentric

* At the present time there is no generally accepted name for the unit of reluctance. Formerly, it was called an oersted but since this term by international agreement has been assigned to the unit of magnetizing force H, see Eq. (5.3), the first three letters of the word reluctance, "rel," will be used in this book to specify the unit of reluctance.

circles around the conductor as shown in Fig. 5.4*a*. The north pole of an exploring compass laid on the cardboard will point in the direction of the magnetic field. The direction of a *magnetic field* is defined as *that direction in which a north pole is urged.* Now, if two conductors carrying current in opposite directions are parallel, the magnetic field will be distorted as shown in Fig. 5.4*b*. It will be noted that for the current flowing out of the paper, the magnetic field is counterclockwise, while for the current flowing into the paper, the magnetic field is clockwise.



(a) Iron filings and compass show presence of circular magnetic field about conductor

(b) Magnetic field about two conductors carrying current in opposite directions

(c) Magnetic field about a single loop of wire

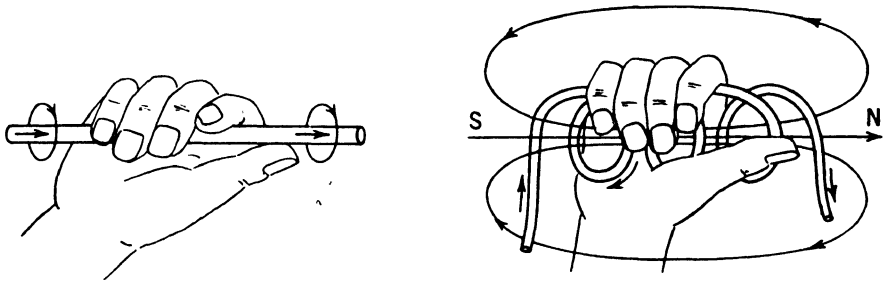
(d) Magnetic field about a coil of wire

FIG. 5.4. Magnetic field around conductors.

If a conductor is formed into a loop, the magnetic field will form around it as shown in Fig. 5.4*c*. Now, if a number of turns are formed to make a coil, the magnetic field will be increased through the center of the coil. The magnetic lines of force circling each loop of the conductor will move in opposite directions between each loop of the coil, and consequently will cancel each other. The result is as if lines of force circling a loop of conductor at one end of the coil continue through the entire coil and emerge from the other end. Since the magnetic lines leave the left-hand end of the coil in Fig. 5.4*d*, it has the properties of the north pole of a permanent magnet. The right-hand end of the coil, where the lines of magnetic flux enter the coil, has the properties of a south magnetic pole. In other

words, the north pole of a permanent magnet would seek the south-pole end of the coil.

5.12. Right-hand Rule. The best way to determine the direction of the magnetic field about a conductor, or the direction of current in a coil of wire, is by the right-hand rule. If a conductor is grasped in the right hand with the thumb pointing in the direction of the current, the fingers will point in the direction of the magnetic field. Similarly, if the direction of the magnetic field is known, the direction of the current can



(a) With conductor in right hand and thumb in direction of current, fingers point in direction of field.

(b) With coil in right hand and thumb in direction of north pole of coil, fingers point in direction of the current in the coil.

FIG. 5.5. Right-hand rule for determining direction of current and field.

be determined. As shown in Fig. 5.5*a*, if the fingers are placed around the conductor in the direction of the magnetic field, then the thumb will point in the direction of the current.

The polarity of a coil of wire can be found if the direction of the current in the coil is known. By the right-hand rule, grasp the coil of wire with the fingers pointing in the direction of the current in the coil; then the thumb will point in the direction of the magnetic field, which is toward the north-pole end of the coil. Similarly, if the polarity of the coil of wire is known, the direction of current flow is readily determined by this rule. Grasp the coil of wire by the right hand so that the thumb will point in the direction of the north pole; then the fingers point in the direction of current flow, as shown in Fig. 5.5*b*.

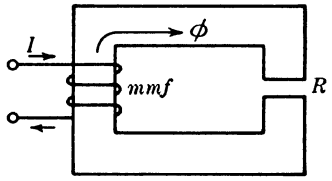
5.13. Air-core and Iron-core Coils. The magnetic properties of a coil depend upon the following factors:

1. Shape and number of turns on the coil
2. Amount of current in the coil
3. Permeability of the core

When the turns are wound loosely, as shown in Fig. 5.4*d*, there is considerable *leakage flux*, that is, lines of force that do not link all the turns in the coil. When the turns are wound closely, most of the flux will link the whole coil. If the number of turns is increased, the magnetic

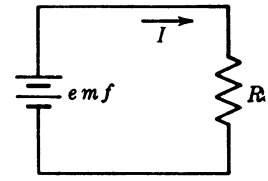
field will be increased for the same value of current. This is because each turn adds its magnetic flux to the total field of the coil.

If the current in the coil is increased, the magnetic field about each turn is strengthened, which results in the total magnetic flux of the coil being made correspondingly greater. If the current in the coil is reversed, the polarity of the coil will also be reversed. The magnetic field produced by a coil of wire carrying current has a shape similar to that of a permanent magnet. The magnetic characteristics of such a coil are similar to those of a permanent magnet.



$mmf = \phi R$

(a) Magnetomotive force produced by coil carrying current forces flux lines around the magnetic circuit.



$mmf = IR$

(b) Electromotive force produced by battery forces current around the electric circuit.

FIG. 5.6. Magnetic and electric circuits.

If a core of soft iron is inserted in the coil of Fig. 5.4*d*, the permeability will be increased, and the magnetic field will be greatly strengthened. This is because the iron core is a better conductor of magnetic lines of force than is air. The iron part of the magnetic path through the coil is quite definite, but the return path through the air outside of the coil is quite indefinite. For a given coil current, the magnetic flux can be further increased by closing the iron path as shown in Fig. 5.6*a*. Then practically all the lines of force will be confined to the iron core linking the coil.

5.14. Comparison of Magnetic and Electric Circuits. Just as electric current is caused to flow in an electric circuit, so can magnetic flux be established in a magnetic circuit. Referring to Fig. 5.6, it will be seen that in many ways these two circuits are similar. Ohm's law applies to the electric circuit of Fig. 5.6*b*. A similar relationship can be applied to the magnetic circuit of Fig. 5.6*a*. A magnetomotive force (mmf) is produced in the magnetic circuit by the current in the coil linking it. This mmf produces a magnetic flux Φ, which is retarded by the reluctance $\mathcal{R}$ in the magnetic circuit. Most of the reluctance $\mathcal{R}$ is across the air gap, because the rest of the magnetic circuit has a high permeability, hence, a very low value of opposition to the magnetic flux.

5.15. Magnetic Circuit Equation. The magnetic flux produced in a magnetic circuit is directly proportional to the mmf and inversely pro-

portional to the reluctance of the circuit. In equation form

$$\text{Magnetic flux} = \frac{\text{magnetomotive force}}{\text{reluctance}} \tag{5.6}$$

By using appropriate symbols, this equation can be written in the three familiar forms as follows:

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}}, \qquad \mathcal{F} = \Phi\mathcal{R}, \qquad \text{and} \qquad \mathcal{R} = \frac{\mathcal{F}}{\Phi} \tag{5.7}$$

where Φ = total number of lines of force, maxwells
$\mathcal{F}$ = mmf, gilberts
$\mathcal{R}$ = reluctance, rel
Now, substituting Φ from Eq. (5.7) in Eq. (5.1),

$$B = \frac{\Phi}{A} = \frac{\mathcal{F}}{\mathcal{R}A} \tag{5.8}$$

Now by using Eq. (5.2) and (5.5) we can write

$$B = \frac{\mathcal{F}}{A}\frac{\mu A}{l} = \frac{\mathcal{F}\mu}{l} \tag{5.9}$$

Since $\mu = 1$ for air

$$B = \frac{\mathcal{F}}{l} = H \text{ for air or nonmagnetic material} \tag{5.10}$$

Example 5.3. A magnetic circuit having a reluctance of 2.5 is supplied with a coil of 1200 turns carrying 0.4 amp. Determine the mmf and magnetic flux.

Solution. By Eq. (5.2)

$$\mathcal{F} = 4\pi NI = (12.566)(1200)(0.04) = 603 \text{ gilberts} \qquad \textit{Ans.}$$

By Eq. (5.7)

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{603}{2.5} = 241.3 \text{ maxwells} \qquad \textit{Ans.}$$

5.16. Magnetic Circuit Laws. It is possible to apply laws to magnetic circuits that are similar to Kirchhoff's laws in electric circuits. In a series magnetic circuit, *the algebraic sum of the magnetomotive forces equals the sum of the reluctance drops.* In equation form

$$\Sigma\mathcal{F} = \Sigma\Phi\mathcal{R} \tag{5.11}$$

where the terms are defined following Eq. (5.7). If branch circuits are being considered, it is well to keep in mind that the *difference in magnetic potential is the same along any path between two points in a magnetic circuit.* This is another way of stating the facts given in Eq. (5.11).

In a parallel magnetic circuit, *the sum of the fluxes toward a given junction*

equals the sum of those leaving the junction. If the algebraic signs are taken into consideration, we can restate this law thus: *The algebraic sum of the fluxes toward a given junction is zero,* or in equation form

$$\Sigma\Phi = 0 \tag{5.12}$$

Example 5.4. A toroid core of cast iron has a cross-sectional area of 0.8 sq in. and a mean length of 25 in. of iron path plus an air gap of $\frac{1}{8}$ in. Assume that the flux

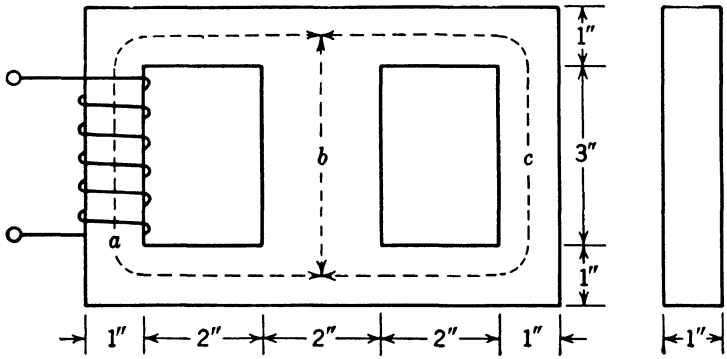


FIG. 5.7. Magnetic circuit with three branches.

density is uniform across the air gap. Determine the total flux when 2 amp flows in a 50-turn coil linking the toroid core, which has a permeability of 300.

Solution. The mmf acting in this series circuit, by Eq. (5.2), is

$$\mathcal{F} = (0.4\pi)(50)(2) = 1{,}257$$

The reluctance of the air gap, by Eq. (5.5), is

$$\mathcal{R}_{\text{air gap}} = \frac{(\frac{1}{8})(2.54)}{(1)(0.8)(6.45)} = 0.0616 \text{ rel} \qquad \text{(in c.g.s. system)}$$

The total reluctance of the series circuit is the sum of the reluctance in the cast-iron core as given in Example 5.2 and the reluctance of the air gap just determined; thus

$$\begin{aligned}\mathcal{R}_{\text{total}} &= \mathcal{R}_{\text{iron core}} + \mathcal{R}_{\text{air gap}} \\ &= 0.0427 + 0.0616 = 0.1043 \text{ rel} \qquad \text{(in c.g.s. system)}\end{aligned}$$

Now, by Eq. (5.11)

$$1{,}257 = 0.0427\Phi + 0.0616\Phi = 0.1043\Phi$$

or

$$\Phi = 12{,}030 \text{ maxwells of flux} \qquad \textit{Ans.}$$

Example 5.5. In the parallel magnetic circuit of Fig. 5.7, the steel has a permeability of 1,000. If leg *a* is wound with a 200-turn coil carrying 1.5 amp, what is the value of flux in the three legs?

Solution. The lengths of legs *a* and *c* are each 11 in. The length of leg *b* is 4 in. The areas of legs *a* and *c* are each 1 sq in., while the area of leg *b* is 2 sq in. The reluctance of leg *c* by Eq. (5.5) is

$$\mathcal{R}_c = \frac{(11)(2.54)}{(1{,}000)(1)(6.45)} = 0.0043307 \text{ rel} \qquad \text{(in c.g.s. units)}$$

By the same formula, the reluctance of the center leg is

$$\mathcal{R}_b = \frac{(4)(2.54)}{(1{,}000)(2)(6.45)} = 0.0007874 \text{ rel} \qquad \text{(in c.g.s. units)}$$

The reluctance of paths b and c in parallel is

$$\mathcal{R}_{bc} = \frac{\mathcal{R}_b\mathcal{R}_c}{\mathcal{R}_b + \mathcal{R}_c} = \frac{(0.0007874)(0.0043307)}{0.0007874 + 0.0043307}$$
$$= 0.00066624 \text{ rel} \qquad \text{(in c.g.s. units)}$$

Since leg a has the same reluctance as leg c, the total reluctance at the coil is

$$\mathcal{R}_{total} = \mathcal{R}_a + \mathcal{R}_{bc} = 0.0043307 + 0.00066624$$
$$= 0.0049969 \text{ rel in c.g.s. units}$$

The magnetomotive force produced by the coil is

$$\mathcal{F} = 0.4\pi NI = (0.4\pi)(200)(1.5)$$
$$= 376.991 \text{ gilberts}$$

Now the flux in leg a is

$$\Phi_a = \frac{\mathcal{F}}{\mathcal{R}_{total}} = \frac{376.991}{0.0049969} = 75{,}444 \text{ maxwells} \qquad \textit{Ans.}$$

The reluctance drop in magnetomotive potential in leg a is

$$\Phi_a\mathcal{R}_a = (75{,}444)(0.0043307) = 326.725 \text{ gilberts}$$

The mmf across legs b and c is then

$$\mathcal{F} - \Phi_a\mathcal{R}_a = 376.991 - 326.725 = 50.266 \text{ gilberts}$$

and the flux in leg b is

$$\Phi_b = \frac{\mathcal{F}_b}{\mathcal{R}_b} = \frac{50.266}{0.0007874} = 63{,}837 \text{ maxwells} \qquad \textit{Ans.}$$

The flux in leg c is

$$\Phi_c = \frac{\mathcal{F}_c}{\mathcal{R}_c} = \frac{50.266}{0.0043307} = 11{,}607 \text{ maxwells} \qquad \textit{Ans.}$$

By Eq. (5.12), the algebraic sum of the fluxes toward the junction is zero; thus

$$\Phi_a - \Phi_b - \Phi_c = 0$$

or the flux toward the junction equals the flux away from the junction; that is,

$$\Phi_a = \Phi_b + \Phi_c$$

Substituting values

$$75{,}444 = 63{,}837 + 11{,}607 \qquad \textit{Check.}$$

It will be noted that in the solution of this series-parallel magnetic circuit, a procedure similar to that employed in solving a series-parallel electric circuit problem was used.

5.17. Magnetic Leakage and Shielding. In electric circuits, the current follows the wire because it is a good conductor, and does not flow through the insulation because it is a poor conductor. The conductivity of copper is usually millions of times better than the surrounding

insulation. In magnetic circuits, the flux follows the iron paths because of its high permeability compared to air or nonmagnetic materials; however, the permeability is only of the order of thousands of times more than that of nonmagnetic materials. It should, therefore, be expected that the leakage flux is high in magnetic circuits as compared to leakage current in electric circuits.

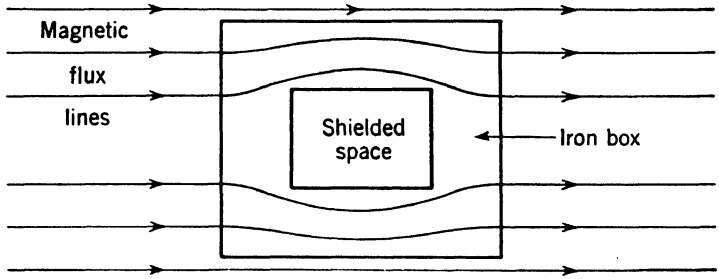


FIG. 5.8. Iron box used as a magnetic shield.

As pointed out before, magnetic circuit calculations do not have the accuracy of electric circuit calculations because of variable permeability. The accuracy is still further decreased by the approximations necessary in estimating the area and length of the various portions of the magnetic path. The area and cross sections of the leakage paths in air are particularly difficult to calculate because of their varying dimensions.

In some applications stray magnetic fields may be very bothersome, causing unwanted effects in electrical apparatus. In such cases the electrical apparatus can be magnetically shielded by placing it in a soft-iron box as shown in Fig. 5.8. Most of the flux in the magnetic field will stay in the metal sides of the box. If this is not sufficient, one iron box can be placed within another iron box. Often copper shields are used in conjunction with magnetic shields to improve the magnetic shielding as well as give electrostatic shielding.

5.18. Magnetic Field Mapping. The accuracy of magnetic circuit calculations can be improved if the shape of the magnetic field is known. In magnetic circuits having irregular boundaries, the magnetic field can be closely approximated by using a graphical method. In plotting a magnetic field, it is important to keep the following facts in mind:

1. *Flux lines are everywhere perpendicular to equipotential surfaces.* In the simple case of a conductor carrying current, as shown in Fig. 5.3, the circular flux lines are at right angles to the radial equipotential mmf surfaces. This reasoning can be readily applied to more complicated cases such as the air gap in Fig. 5.9.

2. *Flux lines, outside of exciting coils, are distributed to minimize the total reluctance.* The flux lines around the air gap of Fig. 5.9, for illus-

tration, show the fringing effect, which minimizes the total reluctance. The lines a between the pole pieces are spaced uniformly and are of equal length because the reluctance is constant. Line b at the edge of the air gap is slightly longer, and since the reluctance is directly proportional to the length of the path, the reluctance of this path must be slightly greater. Since there is a constant mmf potential between the pole pieces, the number of flux lines must decrease. This can be seen by referring to Eq. (5.6) or Eq. (5.7). If $\mathfrak{F}$ is constant and $\mathfrak{R}$ increases, then Φ must decrease. Now, progressing successively to lines c, d, and e, it will be noted that as the lines become longer, the flux density decreases proportionally. For example, if flux line c is twice as long as a, the reluctance along the path doubles and the flux density drops to one-half.

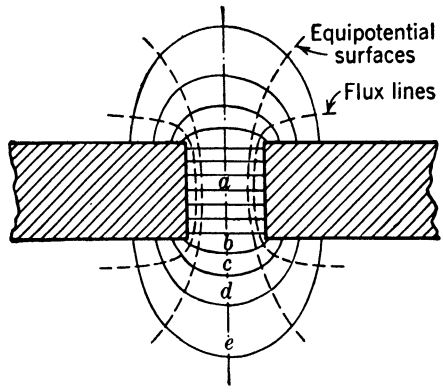


FIG. 5.9. Fringing of flux lines around an air gap.

3. *Flux lines change directions abruptly at surfaces between materials of different permeabilities.* This relationship is given quantitatively by the equation

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_2}{\mu_1} \tag{5.13}$$

where α_1 and α_2 are the angles the flux lines make with the boundary surface between materials having permeabilities of μ_1 and μ_2, respectively, as shown in Fig. 5.10. For illustration, flux lines in air at the surface of iron are nearly perpendicular to the iron because of the large difference in the permeabilities between air and iron. In plotting the flux lines about the air gap in Fig. 5.9, for illustration, they can be drawn perpendicular to the surface of the pole pieces.

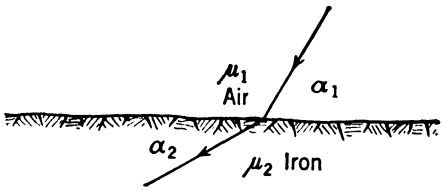


FIG. 5.10. Refraction of flux line at surface.

Example 5.6. A flux line in iron makes an angle of 1° with the surface. If the iron has a permeability of 5,000, what angle in air does the flux line make with the surface?

Solution. Substituting in Eq. (5.13),

$$\tan \alpha_1 = \frac{5000}{1} \tan 1° = 87.275° \qquad \textit{Ans.}$$

A useful approximation, which is reasonably accurate for short air gaps, when determining the area of air gaps, is to add the length of the gap to each cross-section dimension. The flux density, if considered to be uniform in this larger area, will then take care of the fringing effects.

Example 5.7. Determine the reluctance of an air gap in an iron core having a cross section of 1 by 2 in. and a length of 0.2 in.

Solution. By Eq. (5.5),

$$\mathcal{R} = \frac{l}{\mu A}$$

where $\mu = 1$ and the area A, to take care of fringing effects, is

$$A = (1 + 0.2)(2 + 0.2) = 2.64 \text{ sq in.}$$

Then

$$\mathcal{R} = \frac{(0.2)(2.54)}{(2.64)(6.45)} = 0.0298 \text{ rel} \quad \text{(in cgs units)} \quad \textit{Ans.}$$

5.19. Hysteresis Loss. *Hysteresis* is the property a magnetic material has of retaining its magnetism. When a piece of iron is placed in a d-c magnetic field the flux density will increase due to the molecules in the iron which have magnetic properties lining up with the magnetizing force. After most of them have lined up it may be said that the iron has become *saturated;* hence the magnetization curve is not a straight-line linear relation but a curve such as *Oa* in Fig. 5.11. Now if the magnetizing force H is removed, the iron will retain some of its magnetism due to friction, thus preventing the molecules with magnetic properties from following freely the magnetizing force. Thus, in Fig. 5.11 the curve *ab* is followed instead of *aO* when the magnetizing force is removed. This causes a certain amount of energy to be lost as frictional heat in the iron. This loss is called *hysteresis loss.*

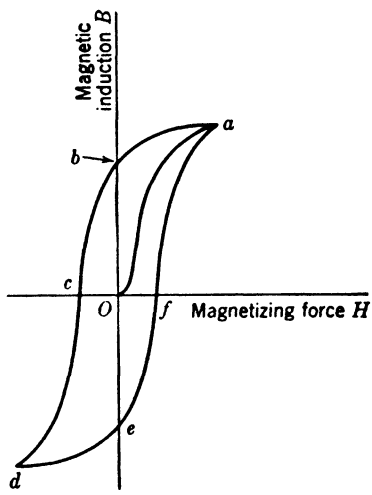


FIG. 5.11. Magnetization curve and hysteresis loop.

If an iron core is magnetized a definite amount, then this magnetism is reversed an equal amount in the opposite direction and the curve tracing the result will be similar to *Oabcd* in Fig. 5.11. If the magnetism is again reversed the curve will return by way of *def* to *a*, thus completing the hysteresis loop *abcdef* to *a*. Each time this magnetic cycle is completed a definite amount of energy is expended. The area of the hysteresis loop is proportional to the hysteresis loss. Steinmetz has developed an empirical

formula to determine the hysteresis loss when the magnetic material is subjected to repeated reversals of magnetism. This fomula can be written

$$P_h = \frac{K_h f V B_{max}^{1.6}}{10^7} \quad \text{watts} \tag{5.14}$$

where P_h = hysteresis loss, watts
K_h = a constant depending upon the quality of the iron. For 4.25 per cent silicon steel this value is 5.35×10^{-4}.
f = frequency, cps
V = volume of solid iron, cu cm
B_{max} = maximum flux density, gausses

Example 5.8. A 4.25 per cent silicon steel core has a flux density of 10,000 gausses. If the core volume is 66.8 cu cm and the frequency is 60 cps, determine the hysteresis loss.

Solution. By Eq. (5.14)

$$P_h = \frac{(5.35)(10^{-4})(60)(66.8)(10{,}000^{1.6})}{10^7} = 0.539 \text{ watts} \qquad \textit{Ans.}$$

5.20. Eddy Current Loss. When current varies in the primary circuit a changing magnetic field causes current to flow in the secondary circuit. If shielding or a transformer core are in this primary field, circulating currents will be developed and the result is a loss of power known as *eddy current loss.* In order to minimize this loss the secondary circuit can be opened or, in the case of shielding, the material can be made into narrow strips connected only at one end. Similarly, transformer cores are laminated and each lamination is insulated from the others. Sometimes this insulation is nothing more than the oxide that forms on the iron. But since the voltage in a thin lamination is very small the insulation need not be large to produce the desired result.

The eddy current loss can be determined by the following equation,

$$P_e = K_e V t^2 f^2 B_{max}^2 \quad \text{watts} \tag{5.15}$$

where P_e = eddy current loss, watts
K_e = a constant depending upon the insulation and flux distribution. For 4.25 per cent silicon steel with good surface insulation this constant is 4.1×10^{-12}.
V = volume of solid iron, cu cm
t = thickness of laminations, cm
f = frequency, cps
B_{max} = maximum flux density, gausses

Example 5.9. With the silicon steel core described in Example 5.8 determine the eddy current loss for laminations having a thickness of 0.0358 cm when the frequency is 60 cps.

Solution. By Eq. (5.15)

$$P_e = (4.1)(10^{-12})(66.8)(0.0358^2)(60^2)(10{,}000^2)$$
$$= 0.1265 \text{ watts} \qquad \textit{Ans.}$$

5.21. Total Core Loss. The total core loss is the sum of the hysteresis and eddy current losses, thus

$$P_c = P_h + P_e \qquad \text{watts} \tag{5.16}$$

where P_c = total core loss, watts
P_h = hysteresis loss, watts
P_e = eddy current loss, watts

Example 5.10. Determine the total core loss for the silicon steel core specified in Examples 5.8 and 5.9.

Solution. By Eq. (5.16)

$$P_c = 0.539 + 0.1265 = 0.6655 \text{ watts} \qquad \textit{Ans}$$

5.22. Skin Effect. Another phenomena of electric currents producing loss is found in large conductors and is known as *skin effect.* Take, for instance, a large stranded conductor that is actually made up of a number of separate wires. The inductance of the various wires in this conductor will be different because the inner wires have more lines of flux linking them than have the outer wires. The result is that a greater counter emf is found at the center of the conductor. This higher reactance at the center causes the current to flow at the surface and hence receives the name "skin effect."

Since the self-inductance causing the counter emf is a direct function of the rate of cutting the lines of flux, it will be greater with higher frequency. It also becomes greater with larger cross sections, greater conductivity, and greater permeability of the conductor.

TABLE 5.1. REDUCTION OF QUANTITY IN C.G.S. UNITS TO M.K.S. UNITS

Quantity	Symbol	c.g.s.	m.k.s.
Magnetic field intensity	H	H oersted	$\left(\frac{10^3}{4\pi}\right) H$ amp turns/meter
Magnetic flux	Φ	Φ maxwell	$10^{-8}\Phi$ weber
Magnetic induction	B	B gauss	$10^{-4}B$ weber/meter;
Magnetomotive force	$\mathfrak{F}$	$\mathfrak{F}$ Gilbert	$\left(\frac{10}{4\pi}\right)\mathfrak{F}$ amp-turn
Permeability	μ	μ gauss/oersted	$4\pi 10^{-7}\ \mu$ henry/meter
Reluctance	$\mathfrak{R}$	$\mathfrak{R}$ rel or gilbert/maxwell	$\left(\frac{10^9}{4\pi}\right)\mathfrak{R}$ amp-turn/weber

At radio frequencies, skin effect causes practically all the current to flow near the surface. This increases the resistance of the conductor.

In some cases of large conductors the current actually flows in the opposite direction at the center of the conductor, due to this counter emf. In most radio frequency circuits it is desirable to minimize this effect by using copper tubing. This lowers the resistance of many such circuits very appreciably. Litz wire is also quite commonly used to wind coils where low loss is of paramount importance. This wire is made up of insulated strands woven in such a way that the inner strands keep coming to the surface at regular intervals.

Summary

Magnets are bodies that possess the property of attracting certain substances such as iron filings.

Permanent magnets retain their magnetism indefinitely.

The direction of a line of force is defined as the direction towards which it will urge the north pole of a magnet.

Lines of force *leave* the *north pole* and *enter* the *south pole.*

Lines of force are *continuous* throughout a magnetic circuit.

Lines of force are *vector quantities* having both magnitude and direction.

Unlike poles attract due to the shortening effect of the lines of force between the poles.

Like poles repel due to the lateral crowding effect of the lines of force around the poles.

Magnetic saturation is reached when an increase of the field does not increase the magnetic properties of the material.

One maxwell of flux is one line of force.

One gauss of flux density is one line of force per sq cm.

$$B = \frac{\Phi}{A}$$

The magnetomotive force produced by one abamp flowing around one turn is 4π gilberts

$$\mathfrak{F} = 4\pi NI$$

When I is in amperes

$$\mathfrak{F} = 0.4\pi NI$$

One oersted of magnetizing force is one gilbert per sq cm.

$$H = \frac{\mathfrak{F}}{l}$$

Permeability is a measure of the ease with which flux can be established in a substance

$$\mu = \frac{B}{H}$$

Reluctance, rel, measured in gilberts per maxwell is the opposition to establishing flux in a magnetic circuit

$$\mathfrak{R} = \frac{l}{\mu A}$$

A continuous magnetic field surrounds a conductor or coil of wire carrying current.

Right-hand rule:

1. With a conductor in the right hand, the fingers will point in the direction of the field when the thumb points in the direction of current flow.
2. With a coil in the right hand, the fingers will point in the direction of current flow when the thumb points toward the north pole.

The magnetic properties of a coil depend upon:

1. Shape and number of turns
2. Amount of current
3. Permeability of the core

The three forms of the magnetic circuit equation are:

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}}; \qquad \mathcal{F} = \Phi\mathcal{R}; \qquad \text{and} \qquad \mathcal{R} = \frac{\mathcal{F}}{\Phi}$$

Field intensity H in lines per sq cm is numerically equal to the magnetizing force in air.

Reluctance in a series circuit can be added just as resistances are added.

$$\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 + \cdots$$

Reluctance in parallel circuits can be added just as resistances are added in parallel circuits

$$\mathcal{R} = \frac{1}{\frac{1}{\mathcal{R}_1} + \frac{1}{\mathcal{R}_2} + \cdots}$$

In a magnetic circuit Kirchhoff's laws are:

1. The algebraic sum of the magnetomotive drops around a complete circuit is zero.
2. The algebraic sum of the fluxes toward a given junction is zero.

All materials conduct magnetic flux.

Leakage flux can be defined as that flux which does not aid in producing the desired effect.

Magnetic shielding does not insulate the space from magnetic flux but shunts the flux around the space.

Flux lines are everywhere perpendicular to equipotential surfaces.

Flux lines outside of exciting coils are distributed so as to minimize the total reluctance.

Flux lines change direction abruptly at surfaces between materials of different permeabilities.

Hysteresis is the property a magnetic material has of retaining its magnetism.

Hysteresis loop is caused by the friction of molecules in a magnetic substance; thus, the magnetic induction does not follow linearly the magnetizing force.

Hysteresis loss is proportional to area of the hysteresis loop. It is also proportional to the frequency, volume of iron, and 1.6 power of the flux density.

Eddy currents are induced currents from varying current in a primary circuit.

Eddy current loss in a transformer core is proportional to volume of iron in field, square of thickness of laminations, square of frequency, and square of flux density.

Total core loss is the sum of hysteresis and eddy current loss.

Exercises

5.1. Two permanent magnets are placed at right angles as shown in Fig. 5.12. Sketch in the magnetic field and indicate the direction of the lines of force.

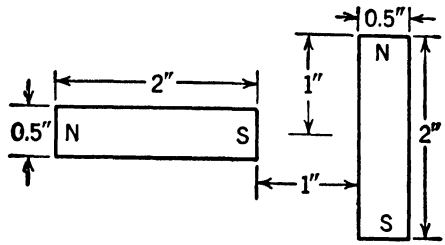


FIG. 5.12. Permanent magnets for Exercise Problems.

5.2. Describe the relationship between current in a straight conductor and the magnetic field associated with it.

5.3. What is the magnetizing force required to produce 100,000 lines of flux in a piece of soft-iron core 2 cm in diameter and having a permeability of 1,200?

5.4. A coil of 75 turns has a current of 2 amp. What is the value of the mmf?

5.5. A series magnetic circuit consists of 10 cm of silicon steel and 6 cm of cast steel. The silicon steel has a permeability of 2,800 and a cross-sectional area of 2.2 sq cms. The cast steel has a permeability of 1,800 and a cross-sectional area of 1.8 sq cms. The total flux is 20,000 lines. Assuming the two metals make perfect contact, what is the value of reluctance and mmf?

5.6. Solve Exercise 5.5 to which an air gap of $\frac{1}{8}$ in. is added to the silicon-steel path and fringing is neglected. What are the results if fringing is not neglected?

5.7. In the parallel magnetic circuit of Fig. 5.13, the annealed sheet steel has a permeability of 2,000. If $a = 32$ in., $b = 8$ in., and $c = 24$ in., what will be the reluctance and flux in each branch if the coil produces 200 ampere turns of mmf? (Neglect leakage flux.)

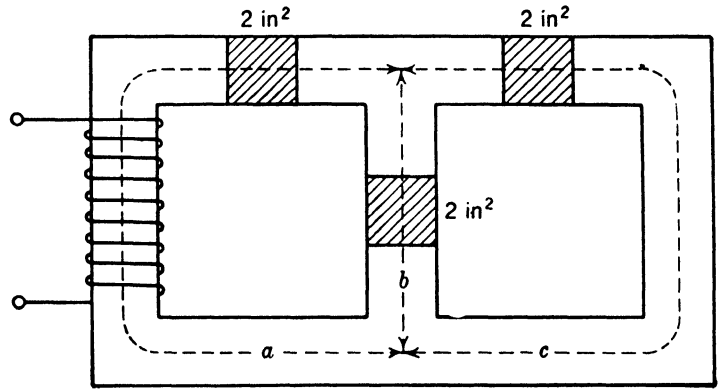


FIG. 5.13. Magnetic circuit with three branches.

5.8. How long an air gap should be cut in branch b of Exercise 5.7 to make the flux in branch a produce 100 kilolines? What will be the flux in paths b and c for this condition? (Neglect leakage and fringing.)

5.9. Determine the air-gap dimensions in Exercise 5.8 if fringing at the air gap is taken into account.

5.10. In a sample of cast iron having a permeability of 200, a flux line makes an angle of 85° with the surface in air. At what angle does the flux line approach the surface in the cast iron?

CHAPTER 6

GENERATOR AND MOTOR ACTION

6.1. Induced Current—A Coil and a Moving Magnet. Oersted's discovery that an electric current flowing through a conductor generated a magnetic field showed that there was a relationship between electricity and magnetism. Many years passed, however, before anyone discovered how a magnet could be used to produce an electric current. Michael Faraday was the first to show that if a magnet is *moved* in the vicinity of a conductor, a voltage will be generated in the conductor. His experiments can be duplicated very easily by connecting a sensitive galvanometer to a coil of wire and suddenly thrusting a bar magnet into the coil; the needle of the galvanometer will be seen to deflect. As soon as the magnet comes to rest, however, the deflection will disappear, showing that it is essential that the magnet be in motion before a voltage can be set up. Sudden withdrawal of the bar magnet from the coil will also cause a deflection in the meter but in the opposite direction to that shown when the magnet was thrust into the coil.

If a current is made to flow through the coil in the same direction as that indicated by the galvanometer when the magnet is thrust into the coil, and the magnetic field around the coil explored by means of a small compass, it will be found that the end of the coil into which the bar magnet is thrust becomes of the same polarity as the end of the bar magnet that is thrust into it. To take a concrete example, if the north pole of the bar magnet is thrust into the coil, the end of the coil that receives the north pole of the bar magnet also becomes a north pole. In other words, the coil pushes the bar magnet away as it tries to enter. Once the bar magnet is inside the coil, however, and then it is suddenly removed, the galvanometer deflects in the other direction, showing that when the magnet is being removed from the coil, the end from which the magnet emerges becomes the south pole. These, and many other phenomena, are examples of Lenz's law, one of the most useful laws in all magnetic theory.

6.2. Lenz's Law. Lenz's law may be stated as follows:

1. *An induced current always opposes the action that produces it.* This same law can be stated in terms of the magnetic field as in (2).

2. *Whenever a change is made in the flux linking a circuit, a voltage is induced or generated in the circuit that tends to oppose the change.*

If a current is permitted to flow in a circuit moving in a magnetic field, it will oppose the mechanical force by motor action, producing the motion. This law will be clarified by the discussion that follows.

6.3. Induced Voltage in a Coil. Further experiments show that neither a moving coil nor a moving magnet is always necessary to induce a voltage in a conductor. A voltage is induced in a conductor when the flux linkage varies, without movement of the conductor or the medium. For example, a voltage e is induced in the secondary winding S of a transformer when the flux linkage varies. The flux linkage varies according to the current I in the primary winding P. It will be noted that neither the conductor, which is the secondary winding, nor the medium, which is the transformer core, has any movement, yet the flux linkages can be varied to induce a voltage in the secondary winding of the transformer.

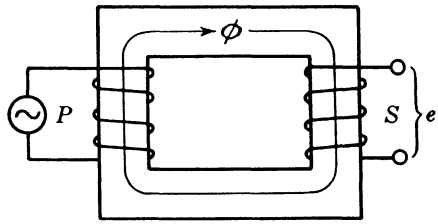


FIG. 6.1. Induced voltage e in the secondary winding S of a transformer due to changing the flux linkage Φ.

If the number of turns in the coil in the transformer experiment is doubled, it will be found that the deflection shown by the galvanometer when the magnet is thrust into the coil will also be doubled. Or if the strength of the bar magnet is doubled, while keeping the number of turns in the coil constant, the deflection will also change. This shows that the voltage induced in a conductor is proportional to the rate of change of flux linkages. Careful measurements repeated many times show that when the flux linking a single turn of conductor is changing at the rate of 10^8 maxwells per sec, an emf of 1 volt will be generated in the conductor. It is not necessary that this change be steady. The voltage at any instant will be proportional to the rate of change of flux linkages. Thus, in general,

$$e = -N \frac{d\Phi}{dt} \times 10^{-8} \tag{6.1}$$

where e = induced voltage, volts
N = number of turns
Φ = flux linkage, maxwells
t = time, sec

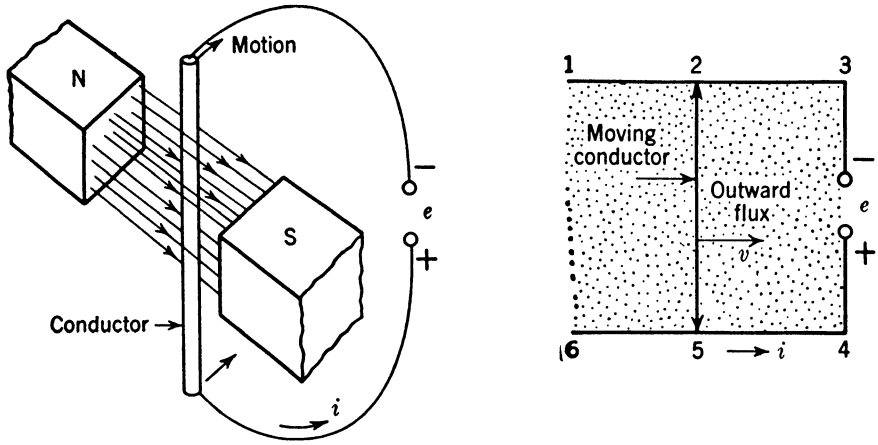
The negative sign results from the fact that the induced voltage tends to *oppose* the change in flux linkage.

One *weber* is defined as that change in flux linkage per sec which will induce 1 volt in a single turn of conductor; thus

$$e = -N \frac{d\Phi}{dt} \tag{6.2}$$

where Φ is the flux linkage in webers and the other terms are as defined in Eq. (6.1).

6.4. Induced Voltage in a Straight Conductor. When a conductor is moved across a magnetic field, as shown in Fig. 6.2, a voltage is *generated.* The amount of voltage generated is proportional to the rate of cutting of the lines of force, or flux linkages. This is the basic principle of the *electric*



(a) Pictorial view of a moving conductor generating a voltage.

(b) Geometrical view to develop formula for generated voltage.

FIG. 6.2. Voltage e is generated when the conductor cuts to the right across the magnetic field.

generator, which is a machine for converting mechanical energy into electrical energy. In some cases the magnetic field is moved while the conductor is held stationary. In either case, a voltage is generated in the circuit that has the flux linkage varying.

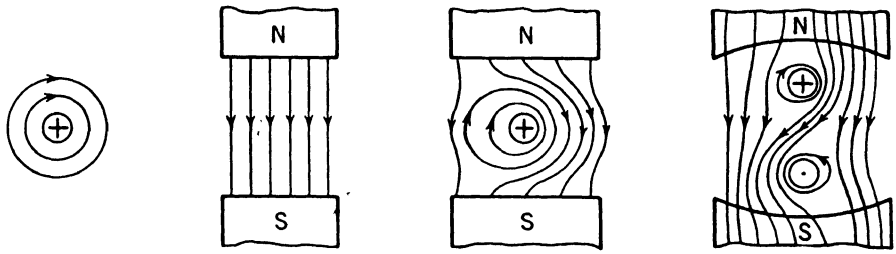
Consider the diagram of Fig. 6.2*b*, which represents a fixed conductor 1-3-4-6 lying in a uniform magnetic field having a flux density of B. If the moving conductor 2-5 has a length l and is moving to the right at a velocity v, the voltage e generated is

$$e = -Blv \tag{6.3}$$

where e = generated voltage, volts
B = flux density, webers/sq m
l = length of conductor, meters
v = velocity of conductor, meters/sec

The negative sign appears because the generated voltage tends to oppose the change in flux linkage. It will also be noted that the terms B, l, and v are vector quantities and must be perpendicular to one another in Eq. (6.3). This equation is a restricted application of the more general

equation as found in Eqs. (6.1) and (6.2). By reference to Fig. 6.2*b* it can be seen that the portion of the two conductors 1-2-3 and 6-5-4 bounded by 2-3-4-5 constitutes a loop through which the flux linkages are changing, because of the movement of the conductor 2-5 to the right through the field. Hence a voltage will be generated in the loop. However, since the only thing that is moving is the conductor 2-5, this voltage can be thought of as being generated entirely by the motion of this conductor through the field, and this concept results in Eq. (6.3).



(a) Field about a conductor carrying current into plane of the paper

(b) A uniform magnetic field

(c) Field about a conductor carrying current in a uniform magnetic field

(d) Field about two conductors carrying current in opposite directions in a uniform magnetic field

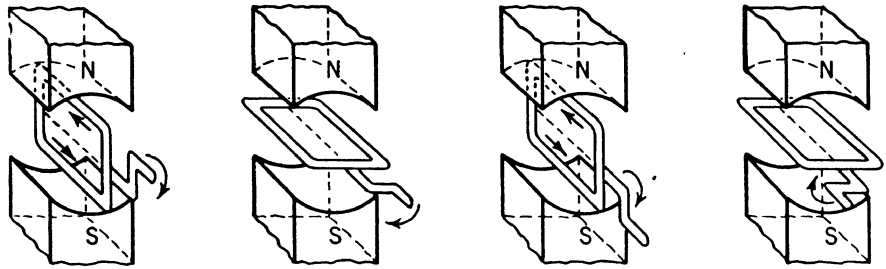
FIG. 6.3. Distortion of a uniform magnetic field with the magnetic field of a conductor carrying current.

When the output terminals of an electric generator are open-circuited as shown in Fig. 6.2, there is no current in the conductor; hence, the magnetic field about the conductor, due to its current, is zero. Under these circumstances, the conductor does not distort the magnetic field, and no energy is required to move it across the magnetic field. When a short-circuited loop of wire is moved through a magnetic field, a voltage is induced in the wire, and as a result of this voltage, current will flow around the loop. This current flowing around the loop sets up its own magnetic field, which is always in such a direction as to oppose the magnetic field producing the current in the loop. To visualize this, refer to Fig. 6.3. In Fig. 6.3*a* we have the undistorted magnetic field about a conductor carrying current. If this conductor is placed in the uniform magnetic field of Fig. 6.3*b*, the distorted magnetic field of Fig. 6.3*c* results. The lines of force are concentrated on the right-hand side of the conductor, and acting like rubber bands, they tend to push the conductor to the left. However, if the conductor is forced to move to the right, the generated voltage tends to cause the current to flow into the plane of the paper as indicated.

The *direction* of the *generated voltage* in a conductor is in the direction of

the *thumb* when the conductor is grasped in the *right hand* and the conductor is moved across the magnetic field in a direction so that the *fingers point* in the direction of the *flux wrapping around the conductor.* In Fig. 6.3*c*, for example, if the thumb points into the plane of the paper, the fingers of the right hand will wrap around the conductor in the same direction as the flux lines wrap around the conductor.

If a loop of conductor is placed between two magnetic poles, as shown in Fig. 6.3*d*, and rotated clockwise, the upper part of the turn will carry current into the plane of the paper and the lower portion will carry current out of the plane of the paper. The magnetic lines of force will stretch around the conductor in such a way as to oppose this motion. This fact is further illustrated in Fig. 6.4, where the complete turn of



(a) Maximum (b) Zero (c) Maximum (d) Zero

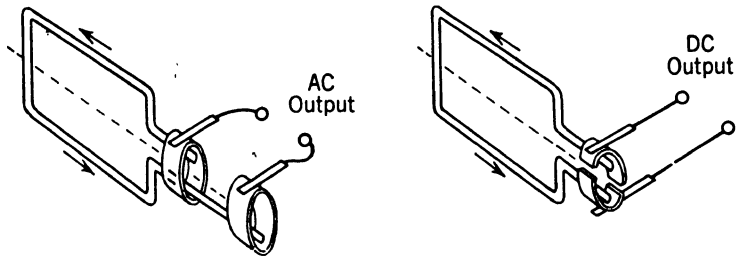
FIG. 6.4. Voltage and current generated in a loop of conductor revolving in a magnetic field.

conductor is shown in perspective. In the position shown in Fig. 6.4(*a*), the loop, if rotated at a uniform rate, will be cutting flux at the maximum rate; hence the maximum voltage and current will be generated in the loop. A quarter turn farther, as shown in Fig. 6.4(*b*), the sides of the loop will be moving parallel to the flux lines, hence cutting no flux and generating no voltage or current. As the coil is rotated to the position shown in Fig. 6.4(*c*), the voltage and current in the loop will again reach a maximum but in the opposite direction through the loop. As the position shown in Fig. 6.4(*d*) is approached, the generated voltage and current again reduce to zero, after which they will reverse direction for the first half of the second revolution.

If two collector rings are connected to the loop, as shown in Fig. 6.5*a*, *alternating current* will be delivered to the output terminals, because for every half revolution, the current in the loop reverses direction. However if the loop is connected to a commutator, as shown in Fig. 6.5*b*, the connections to the output terminals reverse every time the voltage and current generated in the loop reduce to zero. Since both the current in the loop, and the connections to the output terminals, reverse every half

cycle, the output voltage and current will always be in the same direction; hence we have a *d-c generator.*

6.5. Types of D-c Generators. According to the method of excitation, generators may be classified in two general types, namely, (*a*) separately excited, or (*b*) self-excited. The field of a separately excited generator may be a permanent magnet or a field coil receiving current from a storage battery or another generator. Such generators are used when the field must be independent of the terminal voltage or the terminal voltage must be controlled easily, both in magnitude and polarity.



(a) Slip rings deliver alternating current (b) Commutator delivers direct current

FIG. 6.5. Alternating or direct current generated from a revolving loop.

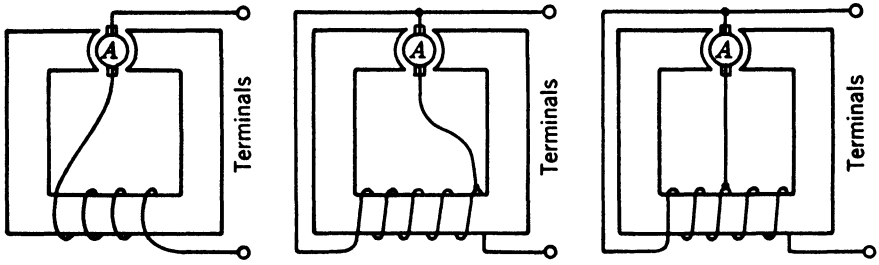
The field of self-excited generators is supplied from the armature of the generator itself. Depending on how the field coil is connected to the armature, a generator is said to be series-wound, shunt-wound, or compound-wound. As the names imply, in a *series-wound* generator the field coil is connected in series with the armature and the output terminals, and in a *shunt-wound* generator, the field coil is connected in parallel with the armature and the output terminals. In a *compound-wound* generator the field coil is connected so that part of it is in series with the armature and load, and part of it is in parallel with the armature and load. The characteristics of these types may be summarized as follows:

1. In the series-wound type, as shown in Fig. 6.6*a*, all the armature current flows through a few field turns. This type produces a terminal voltage and line current that increase as the external load resistance is decreased from an open-circuit condition.

2. In the shunt-wound type, as shown in Fig. 6.6*b*, the field of many turns is connected in parallel with the armature. Only a small portion of the armature current goes through the field winding. The terminal voltage is essentially constant, decreasing slightly with load increases.

3. In the compound-wound generators, as shown in Fig. 6.6*c*, the field consists of both a shunt and a series winding. When the series winding is such that the terminal voltage is constant with changes in load, it is said

to be *flat-compounded.* If the terminal voltage increases with load, it is said to be *overcompounded.*



(a) Series (b) Shunt (c) Compound

FIG. 6.6. Types of d-c self-excited generators or motors.

6.6. Motor Action. Lenz's law applies to both generators and motors. If a current is caused to flow through a conductor in a uniform magnetic field, the field is distorted by an increase in the flux density on one side and a decrease on the opposite side. Since the magnetic lines act like rubber bands, they tend to straighten out and, as shown in Fig. 6.3*c*, will force the conductor to the left. The magnitude of the force is directly proportional to the current in the conductor, the length of the conductor, and the strength of the magnetic field. In equation form

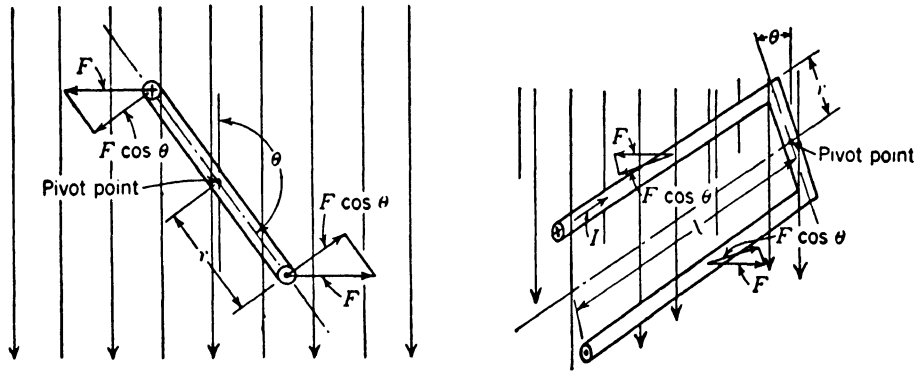
$$F = BlI \tag{6.4}$$

where F = force, newtons
B = flux density, webers/sq meter
l = length of conductor perpendicular to the magnetic field, meters
I = current, amp

A coil of wire carrying current in a magnetic field tends to turn about an axis perpendicular to the magnetic field. In Figs. 6.3*d* and 6.7, the single turn of wire will tend to turn counterclockwise until its plane is perpendicular to the magnetic field. At this point the turning moment about the axis is zero; however, if the coil is moved beyond this position and the direction of the current is reversed, it will continue to rotate 180°. If the current is reversed at correct intervals, the turning moment is always in the same direction. This is accomplished in d-c motors by using commutators.

The magnitude of the turning torque on the coil can be calculated if all the conditions are known. When the plane of the coil makes an angle θ with the direction of the magnetic field, as shown in Fig. 6.7, the coil will tend to turn counterclockwise. This is for the condition that the field is upward and the current is into the plane of the paper for the upper conductor and out of the plane of the paper for the lower conductor.

The force on the upper conductor to the left, by Eq. (6.4), is F, and the force to the right on the lower conductor is also F. The component of



(a) End view (b) Perspective view

FIG. 6.7. A turn of conductor carrying current and lying in a magnetic field is acted upon by a torque, proportional to the cosine of the angle θ, the length of conductor, the flux density, the current, and the length of the lever arm r.

force at right angles to the line joining the conductor to the axis is $F \cos \theta$. Substituting in Eq. (6.4) gives

$$F \cos \theta = BlI \cos \theta \tag{6.5}$$

Since torque is force multiplied by the lever arm r and since there are two conductors of length l involved, we can write

$$T = 2rBlI \cos \theta \tag{6.6}$$

where T = torque, newton-meters
r = length of lever arm, cm

and the other quantities are defined following Eq. (6.4). This equation gives maximum torque when θ is 0 or 180°, which means that the plane of the coil is parallel to the magnetic field; that is, the coil must be in a vertical position as in Fig. 6.7. If the coil is in a horizontal position, $\theta = 90°$ and the torque becomes zero, since the forces are acting in the plane of the coil, leaving no component to turn the coil.

6.7. Types of D-c Motors. As with d-c generators, there are three ways of connecting the field and armature of a motor, namely,

A. Series motors, connected as shown in Fig. 6.6*a*, and having the following characteristics:
1. Decreases speed with increased load
2. Large starting torque
3. High speed at no load (racing)

B. Shunt motors, connected as shown in Fig. 6.6*b*, and having the following characteristics:

1. Nearly a constant speed
2. Low starting torque
3. Speed controlled by controlling field current

C. Compound motors, connected as shown in Fig. 6.6*c*, and having the following characteristics:

1. If connected *differentially* so that the series winding bucks the shunt winding, the motor will have
   a. Shunt characteristics
   b. Nearly a constant speed
   c. Very low starting torque
2. If connected *cumulatively* so that the series winding aids the shunt winding, the motor will have
   a. Series characteristics
   b. Large starting torque
   c. Speed decreases with load increases
   d. Relatively high speed at no load (no racing)

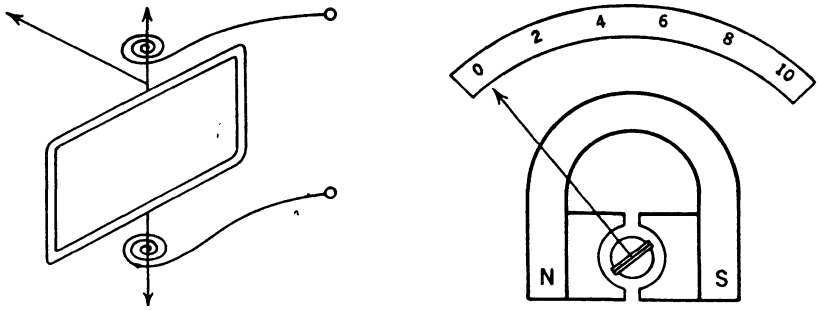
The direction of rotation of a motor, regardless of the type, can be reversed by *reversing* the connections of either the *armature* or the *field* winding, but *not both*. Reversing the direction of the armature current or the magnetic field reverses the direction of the force causing the armature to rotate in the opposite direction. If both the armature current and the magnetic field are reversed, the force vector does not change direction; hence the direction of rotation will remain unchanged.

6.8. Dynamotors and Motor Generator Sets. In radio and communication applications, it is often necessary to convert to another value or kind of current or voltage. A common way of doing this is to use a motor to drive a generator that will deliver the required output. This can be accomplished also by placing two windings on one armature, one acting as a motor and the other acting as a generator. In this way, for example, a low-voltage storage battery can be used to drive the motor, while the generator is wound to deliver a high voltage. Armatures that are to deliver a high voltage at low current are wound with many turns of fine wire, while armatures that are to deliver high current at low voltage are wound with relatively few turns of large wire. Such dynamotors are commonly used for radio sets in automobiles and airplanes. In other applications it may be necessary to convert alternating current to direct current, or vice versa. This can be accomplished by connecting one set of windings to slip rings and the other set of windings to a commutator. Since it is rather difficult to vary the generated voltage output of a dynamotor, the motor generator, which has separate armatures and fields, can be used to advantage for this purpose.

6.9. Electrical Instruments. Many electrical indicating instruments depend upon motor action to move a magnetic material or a conductor

carrying current in a magnetic field. Since instrumentation is so important in electrical circuits, a few of the typical types will be discussed at this time.

a. The D'Arsonval Instrument. This type of instrument is the most popular today for use in *d-c* measurements. It is also widely used in conjunction with thermocouples, copper oxide rectifiers, and vacuum tubes to measure *alternating currents.*



(a) Cut showing moving coil, hair springs, terminals, and pointer.

(b) Cut showing uniform airgap within which the moving coil turns and moves the pointer across the uniformly graduated scale.

FIG. 6.8. Sketch of D'Arsonval-type indicating instrument.

It depends for its operation on the current passing through a coil free to rotate in a uniform magnetic field. The position of the coil is indicated by an attached pointer, as shown in Fig. 6.8*a*, which passes over a scale as shown in Fig. 6.8*b*. The rotation of the coil is limited by the tension of the two hair springs, which obey Hooke's law; thus the deflection is proportional to the current in the coil. The hair springs, as shown in Fig. 6.8*a*, also serve as electrical connections from the meter terminals to the coil of fine wire.

The coil is usually wound over a cylindrical core of soft iron, which serves to increase the magnetic field and make the flux density *uniform* across the air gap (see Fig. 6.8*b*). The torque on the coil is then proportional to the flux density in the air gap and to the current in the moving coil. The scale under the pointer can be graduated linearly to read directly the magnitude of current in the coil. The moving coil-and-pointer assembly is usually attached to a supporting shaft pivoted on jewels, which serve as bearings in addition to being insulators to isolate the terminal connections.

The movement of the coil must be *damped* if the pointer is to come quickly to rest on the correct reading. This can be accomplished by winding the coil on an aluminum frame as shown in Fig. 6.9. This light aluminum frame in addition to being a coil support, is a closed turn of

conductor. When the aluminum frame moves in the magnetic field, it cuts lines of force, which by Lenz's law produces a current in the loop that opposes the motion. This causes the pointer to move more slowly and stop at the final resting position quickly. Without this damping effect,

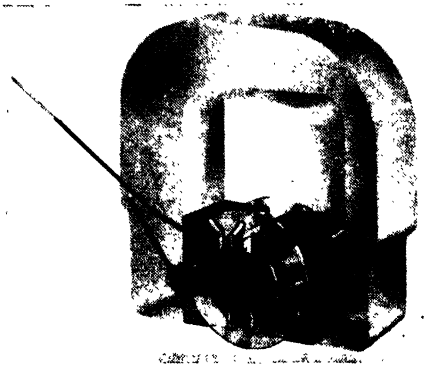
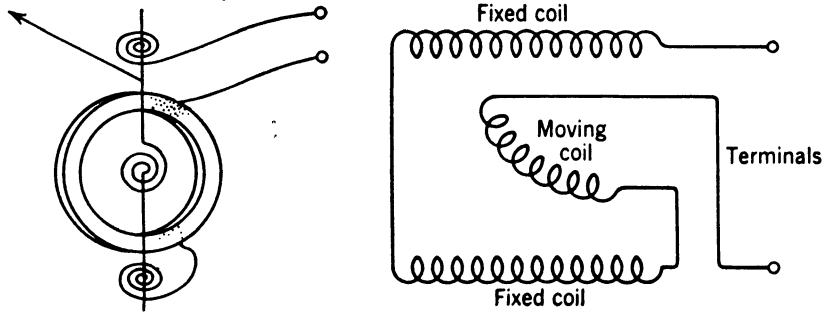


FIG. 6.9. Construction of a D'Arsonval-type indicating instrument. (*Weston.*)

the pointer would overshoot the final reading and swing back and forth until the friction of the air and bearings dissipated the energy.

b. The Electrodynamic Instrument. If the permanent magnetic field is replaced by field coils connected in series with the moving coil, the result is an electrodynamic instrument, as shown in Fig. 6.10. Since a reversal of current reverses both fields, the torque will remain in the same direc-



(a) Operating features (b) Electrical connections

FIG. 6.10. The electrodynamic-type indicating instrument.

tion. Therefore, this type of instrument will operate on alternating current as well as on direct current. This is in contrast to the D'Arsonval instrument, which operates only on direct current. If alternating current were applied to the D'Arsonval instrument, the pointer would be forced first in one direction and then in the other, with the result that it would not deflect at all.

The magnetic field produced by both the fixed and moving coils is

proportional to the current. Since the torque applied to the pointer is proportional to the product of the two fields, the deflection will be proportional to the *current squared.* The scale is therefore crowded at the lower end and expanded at the upper end. Furthermore, it will not be

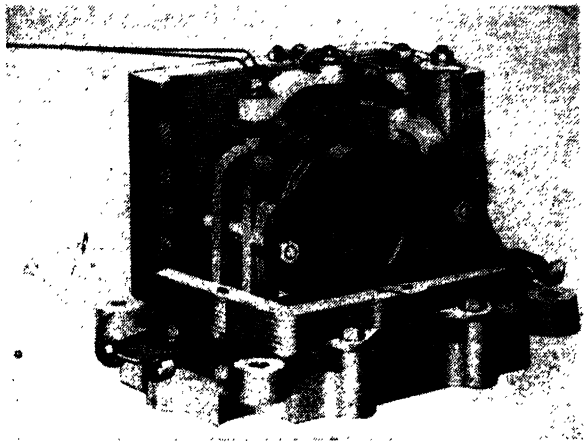


FIG. 6.11. A cut-away view showing details of construction of an electrodynamic-type indicating instrument.

an exact squared scale, because the torque varies also with the position of the moving coil.

Because of the relatively weak magnetic fields, the electrodynamic instrument is usually shielded with laminated iron to prevent errors due to stray fields (see Fig. 6.11). The weak fields also decrease the sensitivity of this type of instrument as compared to the D'Arsonval type which uses a strong permanent magnetic field.

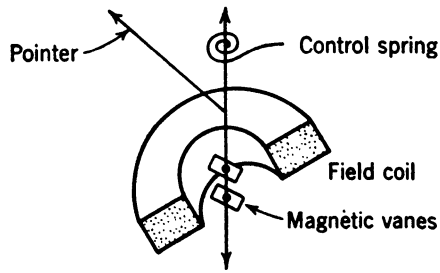


FIG. 6.12. Cut of moving-iron, vane-type instrument.

c. Moving-iron Instruments. Thus far we have considered only the moving-coil type of indicating instruments. The current passing through a fixed coil can be used to attract magnetic material. If the magnetic material is connected to a pointer and restrained by a control spring, an indicating instrument results. Such an arrangement is shown in Fig. 6.12, where the magnetic vanes of soft iron are connected to the shaft carrying the pointer and control spring. There are of course numerous other ways to accomplish this result. In the example given here in Fig. 6.12, current in the field coil produces a magnetic field, which causes the mag-

netic vanes to rotate the shaft and tend to line up with the magnetic field. The deflection is then a function of the current in the field coil. This type of instrument is less sensitive than the D'Arsonval type but has the advantage of operating on either direct or alternating current.

Since the electrodynamic and moving-iron instruments do not have a permanent magnetic field, *damping* must be accomplished by other means

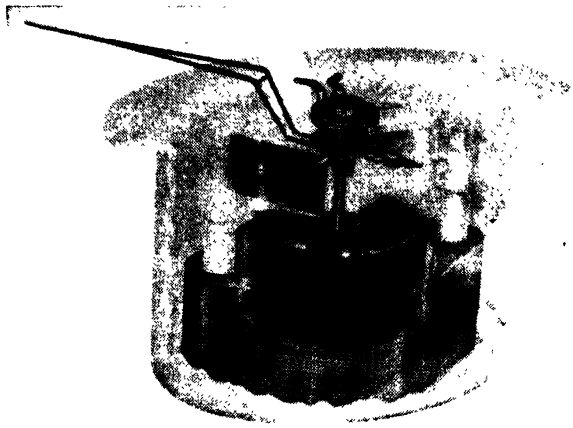


FIG. 6.13. Phantom view of a moving iron instrument.

than the closed loop attached to the rotating shaft. A common and practical way to accomplish damping in a-c instruments is to attach an aluminum air vane to the rotating shaft. If this aluminum air vane is confined to a tight-fitting enclosure, it can be made to operate as a "dashpot" with effective results.

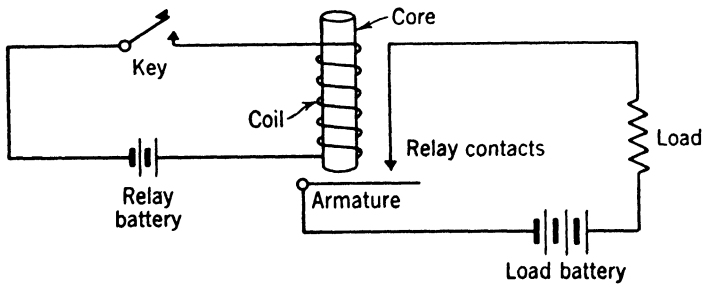


FIG. 6.14. Single-relay circuit.

6.10. Relays. A relay is an electromagnetic device used in general to operate a switch. It is usually a compact sensitive magnetic device used to close or open contactors in another circuit. In many cases it controls the action of large or remote equipment. A very simple relay circuit is shown in Fig. 6.14. When the key is closed, the relay battery causes

current to flow through the relay coil, producing an electromagnetic field in the core; this field attracts the armature, closing the relay contacts and completing the circuit that connects the load battery to the load.

Such a relay is said to be a "normally open" or "n-o" relay. It is also possible for the contacts to be arranged so that a current flowing through the relay coil will open the relay contacts, rather than close them; this type of relay is known as a "normally closed" or "n-c" relay.

The value of current required to move the armature from its normal position is called the *operating current* for a given relay, and the value to which the current must drop in order to release the armature from its operating position is called the *release current.* Both these values can be varied over extremely wide limits by varying the number of turns in the coil, the tension of the spring on the armature, the weight of the armature, the distance through which the armature travels, and the positioning of the device, where the influence of gravity is utilized. The amount of current required to operate a relay is taken as a measure of the *sensitivity* of the relay, and this value may vary from a few microamperes for some types to hundreds of amperes for others.

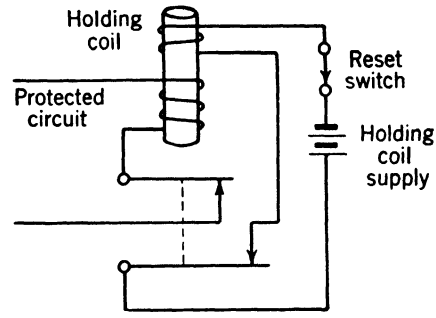


FIG. 6.15. Circuit breaker.

Relays are among the most useful instruments known, because of the wide variety of the tasks they can perform. They can be made to open or close almost any desired number of contacts simultaneously or in a desired sequence. Types of relays are available that will introduce a time delay between the receipt of a signal and the performance of their function. Other types are made that will not operate until a predetermined number of impulses has been received. Yet another type, which is known as a *polarized* relay, will perform one operation when current flows through the coil in one direction, and another operation when the current flow is reversed. With the exception of the latter, which requires direct current for its functioning, all relays can be made to operate on either direct or alternating current, as desired, by the proper design.

One very useful form of relay is the *circuit breaker,* shown in Fig. 6.15. In this relay the contacts that are operated by the armature are in series with the relay coil itself. When the current through the coil exceeds a certain value, the relay operates, opening the circuit. Either mechanical or electrical means are provided for holding the contacts open until they are reset by the operator. In the relay shown, this function is performed

electrically by means of the auxiliary winding, which is called a *holding coil.* After the relay operates, the holding coil maintains the armature in its operated position until the "reset" switch is opened.

Circuit breakers are used extensively in radio transmitters to prevent damage to large tubes from excessive current flow. If, for example, the bias source for the final amplifier in a transmitter became inoperative, very high currents would flow in the plate circuit. By placing a circuit breaker in series with this circuit and the power supply, excessive currents are prevented from flowing for more than a fraction of a second, and no irreparable damage will be done to the tubes.

It is obviously not practical in a chapter of this scope to discuss all the various types of relays in detail, but a few words may be devoted to one form of relay that is of particular interest here, since it furnishes an excellent illustration of the practical application of Lenz's law. This is the *slug-type* time-delay relay shown in Fig. 6.16. In addition to the regular current winding, the core of this relay carries a heavy copper slug, which forms a single turn of very low resistance around the core. When current is suddenly passed through the operating coil, it generates a suddenly increasing magnetic field in the core. This changing flux links the single turn formed by the heavy slug, generating a voltage around the loop and causing a current to flow. This current will itself generate a magnetic field, and by Lenz's law, the direction of this field must be such as to oppose any change in the field producing it.

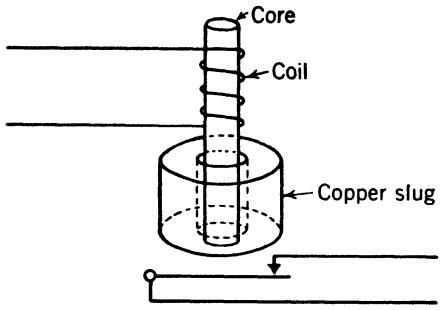


FIG. 6.16. Slug-type time-delay relay.

Since the generating field is increasing, the direction of the induced field must be such as to prevent it from increasing. By making the resistance of the single turn formed by the slug very low, and by positioning the slug on the core in such a manner that practically all the generating flux links this single turn, the counter field set up by the slug can be made so large that the net flux in the core changes from zero only very slowly. Consequently, an appreciable interval must elapse between the time a current is applied to the operating coil and the time the net flux through the core is large enough to operate the armature. Delays up to several seconds can be effected by this means.

Summary

Lenz's law: (1) An induced current always opposes the action that produces it. (2) Whenever a change is made in the flux linking a circuit, a voltage is induced or generated in the circuit that tends to oppose the change.

When the flux linking a coil changes, a voltage is induced in the coil that is given by

$$e = -N \frac{d\Phi}{dt} \times 10^{-8}$$

where e = induced voltage, volts
N = number of turns
Φ = flux linkage, maxwells
t = time, sec

One weber equals the change in flux linkage per second that will induce 1 volt in a single turn of conductor.

When a conductor cuts across a magnetic field, a voltage is generated in the conductor that is given by

$$e = -Blv$$

where e = generated voltage, volts
B = flux density, webers/sq meter
l = length of conductor, meters
v = velocity of conductor, meters/sec

Generator rule: The direction of the generated voltage in a conductor is in the direction of the thumb when the conductor is grasped in the right hand and the conductor is moved across the magnetic field in a direction so that the fingers point in the direction of the flux wrapping around the conductor.

A *separately excited* d-c generator is one in which the field coils are supplied from an external source.

A *self-excited* d-c generator is one in which the generator supplies the current for its own field coils.

A *series-wound* generator or motor is one in which the armature windings and field windings are connected in series.

A *shunt-wound* d-c generator or motor is one in which the armature winding and field winding are in parallel.

A *compound-wound* d-c motor or generator is one in which the field winding consists of both a shunt and a series winding.

When a conductor carrying a current lies in a magnetic field, a force is exerted on the conductor that is given by

$$F = BlI$$

where F = force, newtons
B = flux density, webers/sq meter
l = length of conductor perpendicular to the magnetic field, meters
I = current, amp

Dynamotors and *motor generators* are devices for transforming alternating current to direct current, or vice versa. In a dynamotor the motor and generator coils are wound on a common armature, while in a motor generator the motor and generator coils are wound on separate armatures, but on the same shaft.

A *D'Arsonval-meter* movement consists of a spring-suspended coil lying in the magnetic field of a permanent magnet. A pointer is attached to the coil, and current flowing through the coil causes the coil to move.

An *electrodynamic-meter* movement consists of two coils, spring-mounted at an angle with respect to each other and connected in series. Current flowing through the coils causes the angle between them to change.

A *moving-iron-meter* movement consists of magnetic vanes, spring-mounted at an angle to a field coil and bearing a pointer. Current flowing through the field coil causes the vanes to move.

A *relay* is an electromagnetic device consisting, in essence, of a coil that attracts an armature when current flows through the coil. The movement of the armature is usually made to close or open a switch.

A *circuit breaker* is a relay that opens a circuit when the current through the relay coil exceeds a certain value.

Exercises

6.1. An open-circuited coil of 100 turns lies in a magnetic field. If the flux threading the coil is changing at the rate of 1,000 maxwells/sec, what voltage will appear across the terminals of the coil?

6.2. Is it necessary for a conductor to move relative to a magnetic field in order to generate a voltage?

6.3. A conductor 1 cm in length moves through a uniform magnetic field of density 1 weber/sq meter, at a velocity of 10 meters/sec. What voltage is generated in the conductor if it cuts the flux lines at right angles?

6.4. Define (*a*) series-wound generator; (*b*) shunt-wound generator.

6.5. What is the difference between a dynamotor and a motor generator?

6.6. What is the purpose of the cylindrical iron core between the pole pieces in a D'Arsonval-meter movement?

6.7. Why is the coil of a D'Arsonval-meter movement wound on a conducting frame?

6.8. Name at least three types of electrical instruments that are used for measuring alternating current.

6.9. How would you specify the sensitivity of a relay?

6.10. What is a circuit breaker?

CHAPTER 7

INDUCTANCE

In Chaps. 5 and 6 the voltage induced in a coil lying in a magnetic field was discussed, and it was shown that this voltage was a function of the rate at which the field was changing. Lenz's law was also developed; it states that the induced voltage is always of such a nature that currents flowing as a result of this voltage will oppose any change in the magnetic field producing them, or will oppose the motion of the field and the coil relative to each other. This law follows from the basic law of the conservation of energy.

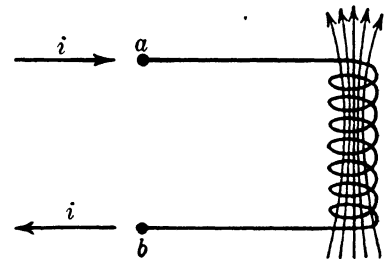


FIG. 7.1. Coil carrying a steady circuit.

No attempt was made, however, to apply this rule to the case where the inducing magnetic field was due to a current flowing through the coil itself. The phenomena that occur in this situation are a result of the property of a coil that is known as *inductance*, which will be the subject of this chapter.

7.1. Coils Carrying Changing Currents. Consider the situation shown in Fig. 7.1. A current i enters the coil at terminal a and leaves at terminal b. This current sets up a magnetic field around the coil. For the moment, let us assume that the wire of which the coil is wound has zero resistance; in other words, the current i is determined by means external to the coil itself.

Under these conditions, if the current i is steady, there will be no voltage difference between the points a and b, since there is no resistance present to cause a voltage drop. A magnetic field exists around the coil, but since the current producing it does not change, the field does not change, and the system is in equilibrium.

Now consider the situation in Fig. 7.2, which is identical with that of Fig. 7.1 except that the current i is increasing. Since the current is increasing, the number of flux lines issuing from the coil must also be increasing. When a *changing* magnetic field exists, Lenz's law applies, and hence a voltage that opposes the change will be generated arcoss the coil. Since the magnetic field is proportional to the current through the coil, this is equivalent to saying that a voltage will be generated that

opposes any change in the current. Since the current is flowing from terminal a to terminal b through the coil, and since the current is increasing, it follows that a voltage will be generated by the coil that opposes the increase by sending current through the coil in the opposite direction, in other words, from b to a. Therefore, the generated voltage will be such that terminal b becomes *negative*, and terminal a becomes *positive*, and this voltage will remain constant as long as i increases at a steady rate.

Now consider Fig. 7.3, where again a current i flows through the coil, except that this current is now decreasing. Because the magnetic field is changing, a voltage that opposes this change will again be generated.

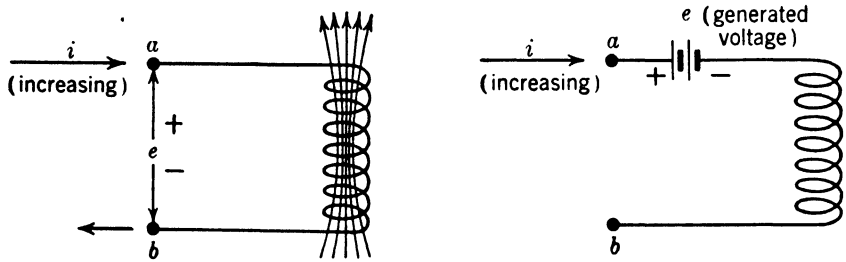


FIG. 7.2. Coil carrying an increasing current.

Since the current is now decreasing, however, the polarity of the voltage generated will be such as to *increase* the flow of current through the coil, or in other words the polarity will be the opposite of that in Fig. 7.2. If the current i decreases at a constant rate, the generated voltage will remain constant.

7.2. Inertia in an Electric Circuit. The effects described in Section 7.1 are exactly analogous to the behavior of a mass m moving at a constant acceleration. In order to accelerate the mass, it is necessary to apply a force to it, and an exactly equal and opposite force will be produced by the mass on the object accelerating it. The velocity of the mass will increase at a constant rate as long as the applied force remains constant.

If for the word "mass" we substitute "coil," for "velocity" we substitute "current," and for "force" we substitute "voltage," statements about masses and coils become interchangeable. This means that if we apply a constant voltage to a coil that has no resistance, the current through the coil will increase at a constant rate as long as the voltage is maintained. Or, to state the facts another way, if the current through a coil changes at a constant rate, a constant voltage will be generated across it.

Just as it is impossible to change the velocity of any mass, however small, from one value to another *instantaneously*, so it is impossible to change the current through a coil, no matter how small, from one value

to another instantaneously. The more quickly the velocity of a given mass is changed from one value to another, the greater the force must be that effects the change. Similarly for a coil, if the current through it must change rapidly from one value to another, a commensurately high voltage must be applied to effect the change.

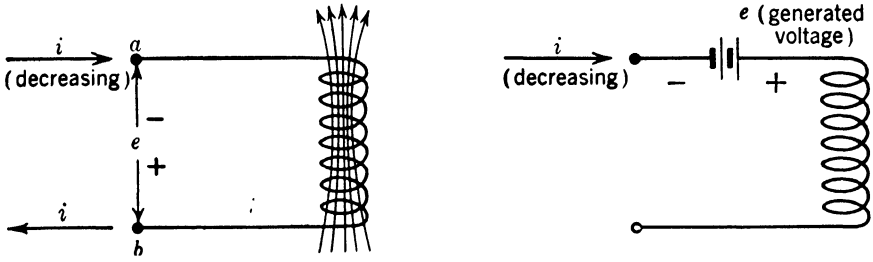


FIG. 7.3. Coil carrying a decreasing current.

If a mass is moving at a constant velocity and is then brought to a sudden stop, a large force will be exerted on the obstacle that stops it. Analogously, if a coil carrying a steady current has the current suddenly stopped, or interrupted, a high voltage will be generated across the terminals of the coil. This is the principle of operation of the ignition coil of an automobile. A switch, or *interrupter*, applies the storage-battery voltage, generally 6 volts, across a coil that has a small spark gap, or *spark plug*, across its terminals. The circuit is held closed long enough for the current through the coil to build up to a fairly high value. The interrupter is then suddenly opened, and the voltage that is generated in the coil, by the sudden interruption of the current through it, is so great that a spark jumps across the spark-plug terminals, igniting the gasoline-air mixture in the cylinders.

7.3. Self-inductance. The property of a coil that causes it to behave as though it had electrical mass is called its *self-inductance*, or, more usually, simply its *inductance*. The unit of inductance is the *henry*, named in honor of Joseph Henry, a scientist who did much important work in magnetism.

A coil has an inductance of one henry when a current through the coil changing at the rate of one ampere per second generates a constant emf of one volt across its terminals.

Example 7.1. What voltage will be generated across the terminals of a coil having an inductance of 8 henrys when the current through the coil is changing at a steady rate of 3 amp/sec?

Solution. The voltage across the coil is proportional to its inductance and to the rate of change of the current through it. Therefore, the generated voltage is given by

$$e = (8)(3) = 24 \text{ volts} \qquad \textit{Ans.}$$

It is not necessary that the current through the coil change at a steady rate. The generated voltage is always proportional to the instantaneous rate of change of the current through the coil. Using the notation of the calculus, the instantaneous rate of change of the current with respect to time can be written as di/dt. The formula for the induced voltage developed across a coil at any instant can therefore be written

$$e = -L\frac{di}{dt} \tag{7.1}$$

where e = instantaneous generated voltage, volts
L = inductance, henrys
di/dt = rate of change of current through the coil, amp/sec

The minus sign in this equation is used because the induced voltage opposes the current change.

If e is constant, this equation can be integrated to derive an expression for the current through an inductance that has a constant voltage applied to its terminals. Rewriting Eq. (7.1), we have

$$di = -\frac{e}{L}\,dt$$

Integrating

$$i = -\frac{e}{L}t + i_0 \tag{7.2}$$

Equation (7.2) shows that the current through an inductance increases linearly with time, is proportional to the applied voltage, and is inversely proportional to the inductance. The integration constant is i_0 the value of current at time $t = 0$.

Example 7.2. What current will be flowing at the end of 3 sec through a coil of 1 henry to which a constant voltage of 6 volts is applied at $t = 0$?

Solution

$$i = -\frac{(6)(3)}{1} = -18 \text{ amp} \qquad \textit{Ans.}$$

Example 7.3. What is the rate of change of current through the coil in the preceding problem?

Solution. The rate of change of the current will be constant and will be given by

$$\frac{di}{dt} = -\frac{e}{L} = -\frac{6}{1} = -6 \text{ amp/sec} \qquad \textit{Ans.}$$

Because the voltage across a coil is proportional to the derivative of the current through it, a coil is referred to in some applications as a *differentiating circuit.*

Figure 7.4 shows two curves. Figure 7.4*a* is the current flowing through a certain coil as a function of time, and Fig. 7.4*b* shows the

voltage developed across the coil by the changing current. The height of the curve in Fig. 7.4*b*, for any value of t, equals the slope of the curve of Fig. 7.4*a* for the same value of t. In other words, Fig. 7.4*b* graphically solves the equation $e = -L\, di/dt$, where i is given by Fig. 7.4*a*.

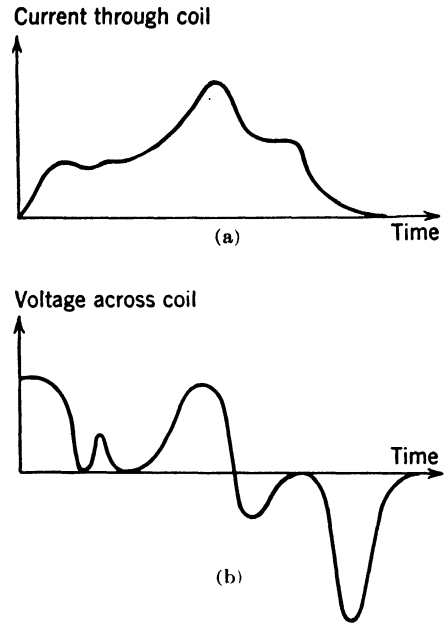


FIG. 7.4. Current through and voltage across a coil.

7.4. The Henry, Millihenry, and Microhenry. The henry is too large a unit of inductance for many applications. For convenience, two other smaller units of inductance are often used, the *millihenry* and the *microhenry*. The millihenry is equal to $\frac{1}{1000}$ henry (0.001 henry) and is commonly abbreviated as mh. The microhenry is equal to 1/1,000,000 henry (0.000001 henry) and is abbreviated as μh.

7.5. Air-core and Iron-core Inductors. It was shown in Eq. (6.1) of the preceding chapter that when a changing flux links a circuit, a voltage is generated that is given by

$$e = -N \frac{d\Phi}{dt} \times 10^{-8} \tag{7.3}$$

where e = instantaneous generated emf, volts
N = number of turns linked by flux
Φ = number of flux lines

This equation obviously applies to the voltage generated by the self-inductance of a coil carrying a changing current. This has been shown to be

$$e = -L \frac{di}{dt} \tag{7.4}$$

Therefore, we can equate these two expressions, and then have

$$L \frac{di}{dt} = N \frac{d\Phi}{dt} \times 10^{-8} \tag{7.5}$$

or

$$L = N \frac{d\Phi}{di} \times 10^{-8} \tag{7.6}$$

which defines L in terms of flux linkages.

If the permeability of the path traversed by the flux lines is constant, then $d\Phi/di$ will be constant, and will equal Φ/i. Since the permeability of air is constant, the inductance of an *air-core* coil (*i.e.*, a coil that contains no iron or similar material) will be independent of the current through the coil. For an air-core coil, the inductance will be given by

$$L = N\frac{\Phi}{i} \times 10^{-8} \tag{7.7}$$

If the coil is wound around a core of iron, or some other ferromagnetic material, however, the number of flux lines produced in the iron will be a complicated function of the current producing the flux. Whenever the permeability of the core is a variable, Eq. (7.6) must be used to calculate L.

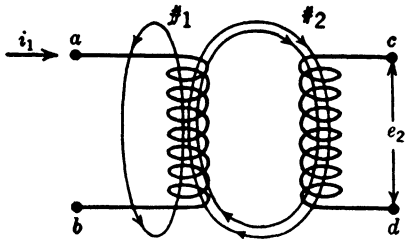


FIG. 7.5. Coils having mutual inductance.

7.6. Mutual Inductance. The Transformer. A voltage will be generated in a coil that is linked by a changing flux, regardless of the source of this flux. Thus, in Fig. 7.5, some of the flux produced by a current i_1 flowing in coil 1 also links the turns of coil 2. If i_1 changes, causing this flux to change, the flux threading coil 2 will also change, and consequently a voltage e_2 will appear across terminals c and d of coil 2. We can therefore write

$$e_2 = -M\frac{di_1}{dt} \tag{7.8}$$

The constant of proportionality, M, in this equation, is called the *coefficient of mutual inductance*, or simply the *mutual inductance*, and is defined as follows:

If a current through one coil, changing at a steady rate of one ampere per second, induces a constant voltage of one volt in another coil, then the two coils are said to have a mutual inductance of one henry.

It should be noted that no matter what the individual inductance of the coils is, the mutual inductance is the same for both. Thus, if two coils have a mutual inductance of M henrys, then a current changing at the rate of n amperes per second in either coil will produce an emf of nM volts in the other coil.

Example 7.4. Two coils have a mutual inductance of 10 mh. If the current through one coil is changing at the rate of 5 ma/sec, what voltage will be generated in the other coil?

Solution

$$e = -(0.01)(0.005) = -0.00005 \text{ volt} \qquad \textit{Ans.}$$

Two coils with mutual inductance constitute what is known as a

transformer. The coil that carries the inducing current is called the *primary*, and the other coil is called the *secondary*. Transformers may take many forms. For r-f work, the primary and secondary may each consist of a few turns of wire spaced an inch or so apart in air, while for a-f work, they may each consist of thousands of turns of wire wound on a common iron core. When the secondary contains more turns than the primary, the result is said to be a *step-up* transformer, and if the secondary contains fewer turns than the primary, it is said to be a *step-down* transformer. Transformers find wide use in a-c work, since, if properly designed, they are one of the most efficient devices known. Furthermore, they have extremely useful impedance-transforming properties which will be discussed in detail farther on in the sections on a-c theory.

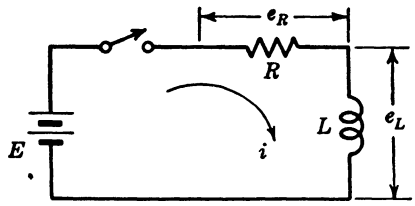


FIG. 7.6. Inductive circuit.

7.7. Switching in an Inductive Circuit. A circuit containing inductance is said to be *inductive*, and the effects obtained by suddenly applying or removing emfs in inductive circuits are of great interest and importance. Consider the situation shown in Fig. 7.6.

Here an emf of E volts can suddenly be applied to a circuit consisting of a resistor R and an inductance L in series. Obviously, from what has been said before, it can be expected that the current I that flows will change with time. It would be natural to expect that it would start at a very low value and gradually increase with time, because of the inertia-like behavior of the inductance.

Since the circuit of Fig. 7.6 is a simple series circuit, two facts may be stated at once: (1) the current through any one element (that is, the battery, resistor, or inductance) at any given instant must equal the current through any other element; (2) the sum of the voltage drops across the resistor and inductance must at all times equal the applied emf E. These are Kirchhoff's laws, and they apply universally to any series circuit.

Let the current i be the current flowing in the circuit at time t. Then the voltage drop across R will be $e_r = Ri$ at time t. The voltage drop across L will be independent of the value of i, however, and will depend only on the rate at which i is changing at time t. In other words the current i varying in the coil causes the change of flux linkages which produces an emf to be set up in the coil having a value

$$e_L = -L\frac{di}{dt} \tag{7.9}$$

This emf of self-induction opposes the change of current in the coil. In other words this emf of self-induction opposes the increase of current when the switch of Fig. 7.6 is closed.

When the switch of Fig. 7.6 is closed the current builds up slowly. For this condition we have the battery potential, the resistance drop, and the emf of self-inductance. Since the two latter are opposed to the battery potential we can write

$$E - Ri - L\frac{di}{dt} = 0 \tag{7.10}$$

which is the differential equation for this circuit. This equation can be re-arranged to give

$$\frac{L\,di}{E - Ri} = dt \tag{7.11}$$

In this form the variables are said to be *separated*, that is, all terms containing i and di are on the left hand side and the dt term is on the right hand side. Each side can now be integrated separately, giving

$$\int \frac{L\,di}{E - Ri} = \int dt$$

or

$$-\frac{L}{R}\ln(E - Ri) = t + \text{constant of integration} \tag{7.12}$$

To determine the constant of integration consider the instant of time $t = 0$ when the switch is closed. At this instant the current through the inductance must be zero, since an infinite voltage would have to exist across L to make the current through it jump instantaneously to some value greater than zero. Therefore, set $t = 0$ and $i = 0$ in Eq. (7.12) and solve for the constant of integration, thus

$$\text{Constant of integration} = -\frac{L}{R}\ln E \tag{7.13}$$

Substituting this value in Eq. (7.12) results in

$$-\frac{L}{R}\ln(E - Ri) = t - \frac{L}{R}\ln E$$

or

$$-\frac{L}{R}\ln\left(\frac{E - Ri}{E}\right) = t$$

or

$$\ln\left(1 - \frac{Ri}{E}\right) = -\frac{Rt}{L} \tag{7.14}$$

Now if $\ln x = y$, it follows that $x = \epsilon^y$; hence, Eq. (7.14) can be rewritten,

$$1 - \frac{Ri}{E} = \epsilon^{-Rt/L}$$

or

$$i = \frac{E}{R}(1 - \epsilon^{-Rt/L}) \tag{7.15}$$

where i = instantaneous current, amp
E = emf, volts
R = resistance, ohms
L = inductance, henrys
t = time, sec

This equation gives the instantaneous value of current in Fig. 7.6 at any time t after the switch is closed. Since t is always a positive quantity, the term $\epsilon^{-Rt/L}$ decreases as t increases, going from a maximum value of 1 when $t = 0$ to a value of zero when t becomes infinite. The maximum value of i will therefore be reached when the time becomes infinite, or in other words i increases continually as t increases, and i approaches but never reaches the limiting value E/R. This value is, of course, the steady value that I would achieve instantaneously if L were not in the circuit. A curve showing i as a function of t is shown in Fig. 7.7.

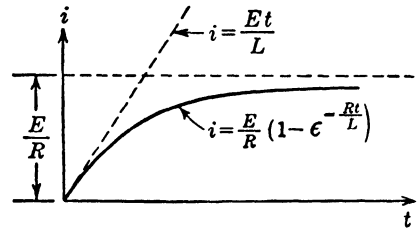


FIG. 7.7. Current transient caused by closing switch in Fig. 7.6.

The dotted line tangent to the curve at the origin shows the current that would have flowed through L as a function of time if R had not been present, and the dotted line parallel to the t-axis shows the current that would have flowed through R if L had not been present.

The current i will rise more slowly for greater values of L. To describe the rate at which the current through an inductance and resistor in series rises, the term *time constant* is used. The time constant T is defined as that value of t for which the exponent of ϵ in Eq. (7.15) becomes equal to -1. Since this exponent equals $(-Rt/L)$, T is given by

$$-\frac{RT}{L} = -1$$

or

$$T = \frac{L}{R} \tag{7.16}$$

When $t = T$, i becomes

$$i = \frac{E}{R}\left(1 - \frac{1}{2.718}\right)$$
$$= 0.632\,\frac{E}{R} \tag{7.17}$$

In other words, the time constant of an inductive circuit is the time required for the current to rise to 63.2 per cent of its final value after a d-c voltage is applied.

Substituting T for L/R in Eq. (7.15) gives another useful form of this equation

$$I = \frac{E}{R}(1 - e^{-t/T}) \tag{7.18}$$

Example 7.5. In Fig. 7.6, let $E = 5$ volts, $R = 10$ ohms, and $L = 2$ henrys. What current will flow in the circuit 0.5 sec after the switch is thrown?

Solution

$$\begin{aligned} i &= \tfrac{5}{10}(1 - e^{-(10)(0.5)/2}) \\ &= \tfrac{1}{2}(1 - e^{-2.5}) \\ &= \tfrac{1}{2}(1 - 0.082) = \tfrac{1}{2}(0.918) \\ &= 0.459 \text{ amp} \end{aligned}$$ *Ans.*

Example 7.6. What is the time constant in Example 7.5?

Solution

$$T = \frac{L}{R} = \frac{2}{10} = 0.2 \text{ sec}$$ *Ans.*

In other words, the current will reach 63.2 per cent of its final value 0.2 sec after the switch is thrown.

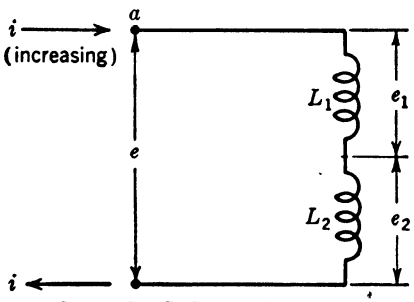


FIG. 7.8. Inductors in series.

7.8. Inductors in Series. A problem that often arises is to find the equivalent inductance of two inductors connected in series. This result can be found very easily by a consideration of Fig. 7.8.

Let the current i flow through L_1 and L_2, and at a given instant let i be increasing at the rate of di/dt amp per sec. This changing current will generate a voltage $L_1(di/dt)$ across the terminals of L_1, and a voltage $L_2(di/dt)$ across the terminals of L_2. The total induced voltage between terminals a and b will then be

$$\begin{aligned} E &= -L_1 \frac{di}{dt} - L_2 \frac{di}{dt} \\ &= -(L_1 + L_2) \frac{di}{dt} \end{aligned} \tag{7.19}$$

This is the same voltage that would be generated across an inductance of $L_1 + L_2$ henrys if a current changing at the same rate flowed through it. We have, therefore: *the equivalent inductance of two or more inductances in series that have no mutual inductance equals the sum of the individual inductances.* If the equivalent inductance be called L, then

$$L = L_1 + L_2 + L_3 + \cdots \tag{7.20}$$

Example 7.7. Inductances of 1, 2, and 5 henrys are connected in series. What is the total inductance?

Solution

$$L = 1 + 2 + 5 \text{ henrys} = 8 \text{ henrys} \qquad \textit{Ans.}$$

7.9. Inductors in Parallel. The equivalent inductance of two inductors in parallel can be determined by a consideration of Fig. 7.9. From Eq. (7.2) the current i_1 through L_1 will induce a voltage e_1 such that

$$i_1 = -\frac{e_1 t}{L_1} \qquad (7.21)$$

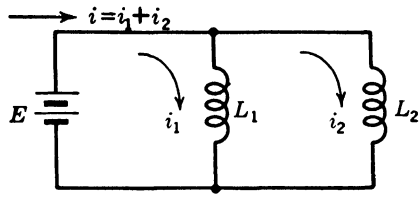


FIG. 7.9. Inductors in parallel.

where t is the time in seconds after the voltage E is applied. The current through L_2 will be given similarly by

$$i_2 = -\frac{e_2 t}{L_2} \qquad (7.22)$$

Now, since the induced voltage opposes the applied voltage we can write

$$E = -e_1 = -e_2 \qquad (7.23)$$

And then the total current flowing from the battery E will be

$$\begin{aligned} i &= i_1 + i_2 \\ &= -\frac{e_1 t}{L_1} - \frac{e_2 t}{L_2} \\ &= \frac{Et}{L_1} + \frac{Et}{L_2} \\ &= \frac{Et}{\dfrac{1}{(1/L_1) + (1/L_2)}} \end{aligned} \qquad (7.24)$$

This is the current that would flow through an inductance of

$$\frac{1}{1/(1/L_1) + (1/L_2)}$$

henrys. We see, therefore, that inductances in parallel behave like resistors in parallel. Hence, the equivalent inductance L of two or more inductances in parallel, and having no mutual effects is given by

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots \qquad (7.24a)$$

where L = total inductance of parallel circuit, henrys

L_1, L_2, L_3, etc. = inductance of branch in parallel circuit, henrys

Example 7.8. Inductances of 1, 2, and 5 henrys are connected in parallel. What is the equivalent inductance of the combination?

Solution. By Eq. (7.25)

$$\frac{1}{L} = \frac{1}{1} + \frac{1}{2} + \frac{1}{5}$$

Therefore $$L = \frac{1}{1.7} = 0.588 \text{ henrys} \qquad \textit{Ans.}$$

7.10. Inductance Calculations. The inductance of a given coil may be computed, using certain formulas that have been worked out, if the dimensions of the coil are known. Only air-core coils will be considered here. As already pointed out, the introduction of iron or other ferromagnetic material in the circuit involves many computational difficulties, which are properly the domain of the specialist. With air-core coils, however, the computations are relatively simple, and of considerable interest.

Three cases will be considered: (1) the single-layer solenoid; (2) the single-layer spiral; and (3) the multilayer solenoid.

7.11. Self-inductance of Single-layer Solenoids. The universal inductance formula is

$$L = an^2F \tag{7.25}$$

where L = inductance
a = radius of coil
n = number of turns
F = constant depending upon the physical dimensions of the coil

If the value of F is plotted from the Lorenz formula[1] for a single-layer solenoid, the curve looks to be hyperbolic in nature and can be practically duplicated if

$$F = \frac{1}{\frac{10b}{a} + 9} \tag{7.26}$$

Substituting this value of F in Eq. (7.25) gives

$$L = \frac{a^2n^2}{9a + 10b} \qquad \mu\text{h} \tag{7.27}$$

where L = inductance, μh (microhenries)
n = number of turns
a = radius of coil, in.
b = length of the winding, in.

Figure 7.10 illustrates these dimensions applied to a single-layer solenoid.

For good coil design the dimensions of the coil should be kept within

[1] This formula is one of the most exact in its field. See E. B. Rosa and F. W Grover, *Bur. Standards Bull.*, Vol. 8, p. 119, 1912.

certain limits. The most inductance can be obtained for a given amount of wire when the ratio of the radius to coil length is approximately $a/b = 1.25$; however, this is not at all critical. This ratio should be kept within the following practical limits:

$$\frac{a}{b} = 0.75 \text{ to } 2.5$$

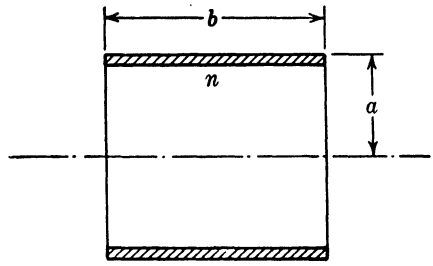


FIG. 7.10. Single-layer solenoid.

Perhaps the easiest way to obtain a given inductance value from Eq. (7.27) is to use the trial-and-error method. That is, select reasonable values of a, b, and n; substitute them in Eq. (7.27), and solve for the inductance L. If the value is too small, increase n or a. If n is increased by adding turns, usually b will increase accordingly. Similarly, if L is too large, use smaller values of a or n.

Example 7.9. A loading coil is needed to load a short antenna. It is desired to design a good coil having an inductance of 20 μh from copper tubing having a diameter of 0.25 in. To conserve copper tubing, let the first trial coil have the ratio of $a/b = 1.25$. The tubing should have a spacing between turn centers of approximately twice its own diameter.

Solution. First Trial. Let

$$a = 5 \text{ in.}, \qquad b = 4 \text{ in.}, \qquad n = 7 \text{ turns}$$

Substituting in Eq. (7.27)

$$L = \frac{(5)^2(7)^2}{(9)(5) + (10)(4)} = \frac{(25)(49)}{45 + 40} = \frac{1{,}225}{85}$$
$$= 14.42 \ \mu\text{h} \qquad \text{(too low)}$$

Second Trial. Let

$$a = 6 \text{ in.}, \qquad b = 5 \text{ in.}, \qquad n = 9 \text{ turns}$$

Substituting in Eq. (7.27)

$$L = \frac{(6)^2(9)^2}{(9)(6) + (10)(5)} = \frac{(36)(81)}{54 + 50} = \frac{2{,}916}{104}$$
$$= 28.05 \ \mu\text{h} \qquad \text{(too high)}$$

Third Trial. Let

$$a = 5.5 \text{ in.}, \qquad b = 4.5 \text{ in.}, \qquad n = 8 \text{ turns}$$

Substituting in Eq. (7.27)

$$L = \frac{(5.5)^2(8)^2}{(9)(5.5) + (10)(4.5)} = \frac{(30.25)(64)}{49.5 + 45} = \frac{1{,}935}{94.5}$$
$$= 20.46 \ \mu\text{h} \qquad \textit{Approximate Ans.}$$

If b, the length of the coil, is increased slightly, the inductance can be made exact. To do this, solve Eq. (7.27) for b as follows: Multiply both sides by the denominator. Thus, multiplying both sides by $9a + 10b$ gives

$$9aL + 10bL = a^2n^2$$

Subtracting $9aL$ from both sides gives

$$10bL = a^2n^2 - 9aL$$

Dividing both sides by $10L$ gives

$$b = \frac{a^2n^2 - 9aL}{10L} = \text{length, in.} \tag{7.28}$$

By substituting in Eq. (7.28) the desired value of inductance, $L = 20$, and the values of a and n used in the third trial above, the length of the coil (l_c in.) can be determined.

$$b = \frac{(5.5)^2(8)^2 - (9)(5.5)(20)}{(10)(20)}$$

$$= \frac{(30.25)(64) - (49.5)(20)}{200} = \frac{1{,}935 - 990}{200}$$

$$= {}^{945}\!/_{200} = 4.725 \text{ in.}$$

This is the length of the coil to give the correct inductance.
Checking,

$$L = \frac{(5.5)^2(8)^2}{(9)(5.5) + (10)(4.725)} = \frac{(30.25)(64)}{49.5 + 47.25} = \frac{1{,}935}{96.75}$$

$$= 20 \ \mu\text{h} \qquad \textit{Exact Ans.}$$

The spacing between turn centers is

$$S = \frac{b}{n} = \frac{4.725}{8} = 0.59 \text{ in.}$$

The following equation can be used to determine the length of copper tubing necessary to construct the coil:

$$l_c = \frac{2\pi an}{12} \quad \text{ft} \tag{7.29}$$

where l_c = length of conductor in coil, ft
$\pi = 3.1416$
a = radius of coil, in.
n = number of turns in coil

Substituting

$$l_c = \frac{(2)(3.1416)(5.5)(8)}{12} = 23 \text{ ft}$$

The coil specifications can now be made as follows:

$l_c = 23$ ft of 0.25-in.-diameter copper tubing
$a = 5.5$ in.-radius of coil
$b = 4.725$ in.-length of coil
$n = 8$ turns
$S = 0.59$ in.-spacing between turn centers
$L = 20$ μh-inductance

When it is not necessary to obtain an exact value of inductance, the spacing S should be left in some dimension easy to measure. Equation (7.27) can be depended on only for an accuracy within about 2 per cent. The final and best way to obtain a good value of inductance is by measure-

ment. Equations as given in this chapter, however, predict quite closely the high-frequency inductance of coils of various shapes. In fact, the inductance can be calculated as closely as the coil can reasonably be constructed from the specifications.

7.12. Self-inductance of Single-layer Spirals. Equation (7.27) can be used to determine the inductance of a spiral if c, the radial thickness of the winding, is substituted for b. Thus

$$L = \frac{a^2n^2}{9a + 10c} \quad \mu h \qquad (7.30)$$

where L = inductance, μh (microhenrys)
n = number of turns
a = mean radius of spiral, in.
c = radial depth of spiral, in.

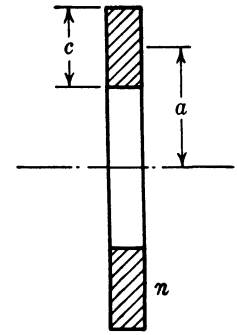


FIG. 7.11. Single-layer spiral.

Figure 7.11 illustrates how these dimensions are applied to a single-layer spiral.

Example 7.10. It is desired to construct a spiral inductance of 15 μh, which is to be used as the primary of an r-f transformer to couple the output of a 1-kw transmitter to the antenna.

Solution. If $\frac{5}{16}$-in. copper tubing is used to make the spiral, the spacing between turn centers should be approximately $\frac{5}{8}$ in. Starting the spiral at a radius of 3 in., a trial calculation can be made as follows:

$$S = \tfrac{5}{8} \text{ in.} = 0.625 \text{ in.} = \text{spacing between turn centers}$$
$$n = 7 \text{ turns}$$
$$c = nS = (7)(0.625) = 4.375 \text{ in.}$$
$$a = \frac{c}{2} + 3 = \frac{4.375}{2} + 3 = 5.1875 \text{ in.}$$

Substituting in Eq. (7.30)

$$L = \frac{(5.1875)^2(7)^2}{(9)(5.1875) + (10)(4.375)} = \frac{(26.9)(49)}{46.69 + 43.75} = \frac{1{,}318}{90.44}$$
$$= 14.57 \ \mu\text{h inductance}$$

This value is very close. If 8 turns are used, the inductance $L = 19.44$ μh. Interpolating for the inductance gives

$$\frac{1 \text{ turn}}{19.44 - 14.57} = \frac{x \text{ turns}}{15 - 14.57}$$

or

$$x \text{ turns} = \frac{15 - 14.57}{19.44 - 14.57} = \frac{0.43}{4.97} = 0.0865$$

This means that for 15 μh inductance

$$n = 7.0865 \text{ turns}$$

This example shows that for practical purposes, 7 turns is close enough. The accuracy of both Eqs. (7.27) and (7.20) may be less than 2 per cent if too few turns are used, or when the spacing between turns is too great.

7.13. Self-inductance of Multilayer Coils. When the dimensions of a multilayer coil are as shown in Fig. 7.12, the inductance is

$$L = \frac{0.8a^2n^2}{6a + 9b + 10c} \quad \mu h \tag{7.31}$$

where L = inductance, μh
a = mean radius, in.
b = width of coil, in.
c = radial depth of coil, in.
n = number of turns in the coil

The accuracy of this equation is within 1 per cent when the three terms in the denominator are approximately equal.

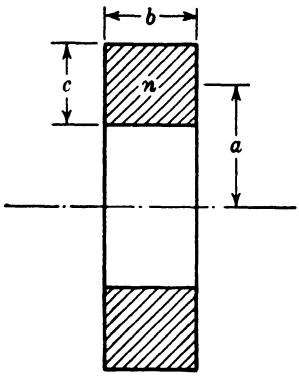


FIG. 7.12. Multilayer coil.

Example 7.11. Find the inductance of a multilayer coil wound with No. 22 dcc copper magnet wire on a spool having dimensions

a = 1.5 in.
b = 1 in.
c = 0.9 in.

Solution. From a wire table we find that this wire can be wound at the rate of 30 turns per in. Then the number of turns per layer will be $30b = 30$, and the number of layers will be $30c = 27$. The number of turns is then $n = (30)(27) = 810$, the total number of turns in the coil. Substituting in Eq. (7.31) gives

$$L = \frac{(0.8)(1.5)^2(810)^2}{(6)(1.5) + (9)(1) + (10)(0.9)} = \frac{(0.8)(2.25)(656{,}100)}{9 + 9 + 9}$$

$$= \frac{1{,}181{,}000}{27} = 43{,}750\ \mu\text{h inductance}$$

or $L = 43.75$ mh inductance *Ans.*

This result should be accurate within 1 per cent since the terms in the denominator are equal.

Summary

Inductance is an electrical quantity that is analogous to mass.

If a changing current flows through an inductance, a voltage will be generated that opposes the change.

The unit of inductance is the *henry*, which is defined as the inductance that will generate a voltage of 1 volt across its terminals when the current through it is changing at the rate of 1 amp/sec.

The *voltage* generated or induced across an inductance is given by

$$e = -L\frac{di}{dt}$$

where e = generated or induced emf, volts
L = inductance, henrys
$\frac{di}{dt}$ = rate of change of current, amp/sec

If a steady voltage is applied to an inductance, a *current* will start to flow through it that is given by

$$i = \frac{Et}{L}$$

where i = instantaneous current, amp
E = applied emf, volts
t = time after emf is applied, sec
L = inductance, henrys

When a current changing at the rate of di/dt amperes per second generates or induces a voltage of $-M di/dt$ volts in another coil, the two coils are said to be coupled by a *mutual inductance* of M henrys.

If an emf E is applied at time $t = 0$ to a circuit consisting of R ohms and L henrys in series, the current will be given by

$$I = \frac{E}{R}(1 - e^{-Rt/L})$$

The *time constant* of an inductive circuit is the time required for the current to rise to 63.2 per cent of its final value after a constant voltage is applied.

$$T = \frac{L}{R}$$

For *inductors in series*, the equivalent inductance is given by

$$L = L_1 + L_2 + L_3 + \cdots$$

For *inductors in parallel* with no mutual effects, the equivalent inductance is given by

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots$$

The *inductance of a single-layer solenoid* is given by

$$L = \frac{a^2n^2}{9a + 10b}$$

where L = inductance, μh
n = number of turns
a = radius of coil, in.
b = length of winding, in.

The *inductance of a single-layer spiral* is given by

$$L = \frac{a^2n^2}{9a + 10c}$$

where L = inductance, μh
n = number of turns
a = mean radius of coil, in.
c = radial depth of spiral, in.

The *inductance of a multilayer solenoid* is given by

$$L = \frac{0.8a^2n^2}{6a + 9b + 10c}$$

where L = inductance, μh
n = number of turns
a = mean radius, in.
b = width of coil, in.
c = radial depth of coil, in.

Exercises

7.1. In an inductive circuit, what is the relationship between the voltage generated and the current that generates it? State in words.

7.2. Inductance is analogous to what mechanical quantity?

7.3. What is the unit of self-inductance?

7.4. What is the formula relating inductance, changing current, and generated voltage?

7.5. What is mutual inductance?

7.6. What is meant by the "time constant" of an inductive circuit?

7.7. If an emf of 10 volts is applied at time $t = 0$ to a circuit consisting of 1 ohm and 1 henry in series, what current will flow in the circuit at the end of 1 sec?

7.8. What is the equivalent inductance of two coils, each of 5 henrys inductance, in series?

7.9. Give the formula for the inductance of a single-layer solenoid.

7.10. What is the inductance of a spiral of 10 turns when the mean radius of the spiral is 2 in. and the radial depth of the spiral is 2 in.?

CHAPTER 8

ELECTROSTATIC CAPACITY

8.1. Capacitors and Dielectrics. A capacitor consists of two conductors separated by an insulating medium, which is called the *dielectric*. Capacitors may take many forms. For example, a very common form of capacitor, known as the *paper capacitor*, consists of two pieces of tin foil separated by a piece of waxed paper. For use in r-f circuits, capacitors are often made up of pieces of tin foil separated by very thin slabs of mica. Capacitors for use in high-voltage circuits are often manufactured with oil as the dielectric. Or, when a-c voltages of high frequency are to be used, the dielectric is sometimes a vacuum.

Variable capacitors are generally made with air as the dielectric, and the capacity is varied by moving two metal plates, or two sets of metal plates, with respect to one another.

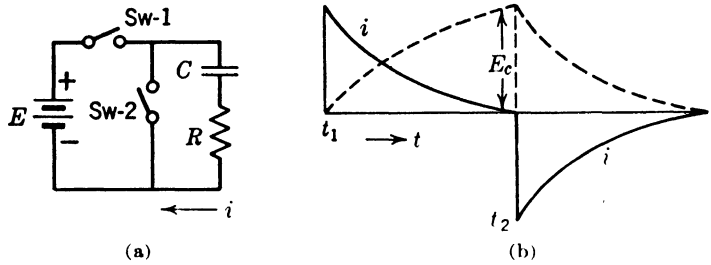
Whatever its form, however, the function of a capacitor is to *store electric charge*. When a voltage is supplied across the plates of a capacitor, electric charge is stored on the plates and will remain there until the capacitor is discharged, either through leakage due to poor resistance of the dielectric, or by completion of an external circuit between the plates of the capacitor. The storage of charge in a capacitor is analogous to the storage of potential energy in a mechanical circuit. This topic will be discussed in the next section.

8.2. Elasticity in Electric Circuits. A capacitor in an electric circuit is in many respects comparable to a spring or compliance in a mechanical system. The mechanical analogy for a capacitor in series with a resistance will follow.

If a voltage E is applied to a circuit containing a capacitor C in series with a resistance R, as shown in Fig. 8.1*a*, a current i will flow, as shown in Fig. 8.1*b*, when the switch 1 is closed at time t_1. The initial current that will flow the instant the switch is closed will be $i = E/R$. This current then gradually decreases and becomes zero. When the switch is first closed, the circuit acts as though the capacitor was not in the circuit, and the current instantly rises to a value depending on the resistance of the circuit. The current causes charges to build up on the plates of the capacitor, until it finally becomes fully charged and the voltage is equal to the battery voltage. The voltage across the capacitor plates is shown by the dotted line in Fig. 8.1*b*. When the voltage across the

capacitor plates becomes equal to the battery voltage, the capacitor is fully charged and no more charging current will flow through the resistor.

Now, open switch 1 and close switch 2 at time t_2, as shown in Fig. 8.1. The voltage across the capacitor will cause the current to flow through the resistance R in the reverse direction. The first instant this current will be $i = E_c/R$, and as before it will gradually decrease to zero when the capacitor has completely lost its charge. The voltage across the



(a) (b)

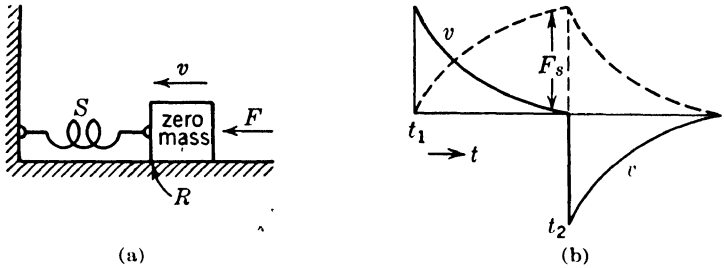
FIG. 8.1. Current in an electric network containing a capacity in series with a resistance.

capacitor will gradually decrease to a zero value as shown by the dotted line in Fig. 8.1*b*. In the second case, the current will flow in the opposite direction, as shown in Fig. 8.1*b* by the negative values.

This circuit in the mechanical system can be represented by a spring fixed at one end, as shown in Fig. 8.2*a*, and tied to a body of zero mass. If the force F is applied at time t_1, the velocity v at the first instant will be limited only by the frictional resistance R, but gradually the spring will become compressed more and offer more opposing force. This will cause the velocity v to decrease gradually to zero as shown in Fig. 8.2*b*. When the movement stops, the force offered by the spring will be exactly equal to the applied force F. Next, at time t_2 remove the force F. Since the spring contains energy owing to its compressed shape, it will expand, and the weightless body will move rapidly at first and then gradually slow down and stop as shown in the graph of Fig. 8.2*b*. It will be noted that the graph of the velocity v of the free end of the spring shown in Fig. 8.2*b* is of exactly the same shape as the graph of the current i in the series circuit shown in Fig. 8.1*b*. The capacitor C of Fig. 8.1*a* becomes charged when a voltage E is applied just as a spring S becomes compressed when a force F is applied. In the case of the capacitor, the electronic orbits are displaced (as shown in Chap. 1, Matter and Energy) when a voltage is applied across the plates of the capacitor. Likewise, when a spring is compressed, a distortion takes place within the atoms of the metal.

It may be well at this point to mention the fact that if the voltage E_L across the inductance L of Fig. 7.6 were plotted, it would be similar to the current i shown in Fig. 8.1*b*. This means that when the switch of

Fig. 7.6 is closed, the battery voltage E is instantly applied across the inductance. As the current begins to flow in the inductance, a counter emf set up across the coil decreases until finally the voltage drop across the coil is zero, if it contains no resistance. Similarly, when the switch is opened, the collapsing magnetic field causes a high voltage at first in the opposite direction, and it gradually dies down to zero again when all the energy of the magnetic field is expended.



(a) (b)

FIG. 8.2. Velocity v of a mechanical system containing a spring and a frictional resistance.

The above example shows that an inductance and a capacitor have exactly opposite characteristics. That is, the current flowing through an inductance is similar to the voltage across the capacitor, and the voltage across an inductance is similar to the displacement current flowing through a capacitor. This is a very important relationship and will be used later in a-c circuit theory.

8.3. The Electrostatic Field. When the capacitor C of Fig. 8.1*a* is charged, the insulating medium, known as a *dielectric*, is in a state of stress. For convenience, it is customary to represent the force causing the stress by lines drawn from the positively charged body to the negatively charged body. These lines are drawn parallel to the acting force and have arrows on the lines pointing in the direction a small positive charge would tend to move if placed at the point in question. Figure 8.3 shows the *electrostatic lines of force* between two spheres with opposite charges. These spheres will experience a force of attraction in this case (unlike charges attract, while like charges repel). This force F will depend upon the sizes of the charges and the distance of separation and, according to *Coulomb's law*, can be written

$$F = \frac{Q_1 Q_2}{d^2} \tag{8.1}$$

where F = force acting between the two charged bodies, dynes
Q_1 and Q_2 = charge on the respective bodies, statcoulombs
d = distance of separation, cm
This equation is true for air or a vacuum.

8.4. Dielectric Constant. In the preceding case, the number of lines of force per unit area is called the *field intensity*. If some other insulating material is used, such as insulating oil, the lines of force are more easily formed. In other words, the number of lines of force per unit area, known as the *flux density*, would increase. Since various insulating mediums have different insulating properties, the term *dielectric constant* is introduced to tell quantitatively how much the lines of force will be increased. The dielectric constant is defined by the following equation:

$$K = \frac{\text{flux density}}{\text{field intensity}} \tag{8.2}$$

It will be noted that the dielectric constant K plays the same role with a dielectric as the permeability μ plays with a magnetic material.

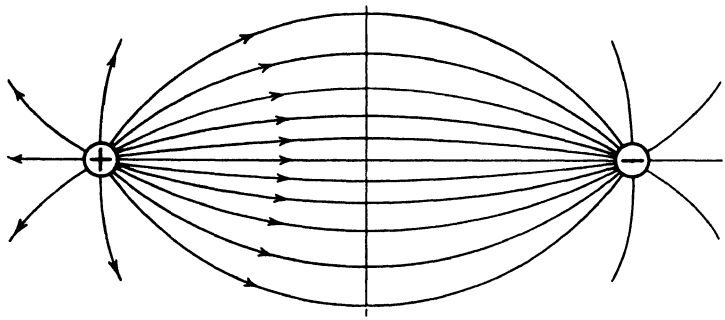


FIG. 8.3. Electrostatic field between two unlike charges.

Coulomb's law for an insulating medium of dielectric constant K can be written

$$F = \frac{Q_1 Q_2}{K d^2} \tag{8.3}$$

where K is the dielectric constant and the other symbols are the same as in Eq. (8.1). Tables of dielectric constants are given in handbooks.

8.5. Electrostatic Potential. If a charge is moved in opposition to an electrostatic force, work is done, and energy is stored in the charge. This energy is in the form of potential energy, just as potential energy is stored in a book (mass) if it is raised from the floor to the top of a table. The book on the table is said to have a certain potential energy due to its position on the table. Similarly, an electric charge will have a certain potential energy due to its position in an electrostatic field. *The potential difference (or voltage) between two points in an electrostatic field is defined as equal to the work required to move a unit positive charge from one point to the other.* Usually, in practice, the change in electrostatic field potential is expressed in volts per inch, which means that it will take a certain

voltage to move a coulomb charge of electricity one inch. In much scientific work, the field gradient is measured in volts per centimeter.

8.6. Equipotential Surfaces. A charge can be moved at right angles to the electrostatic lines of force without doing work or changing its electrostatic potential. For instance, in Fig. 8.4 a charge can be moved along the vertical line between the two charges without doing work (or changing its potential). Similarly, no work will be done on a charge if it is moved around one of the other equipotential lines. However, if a

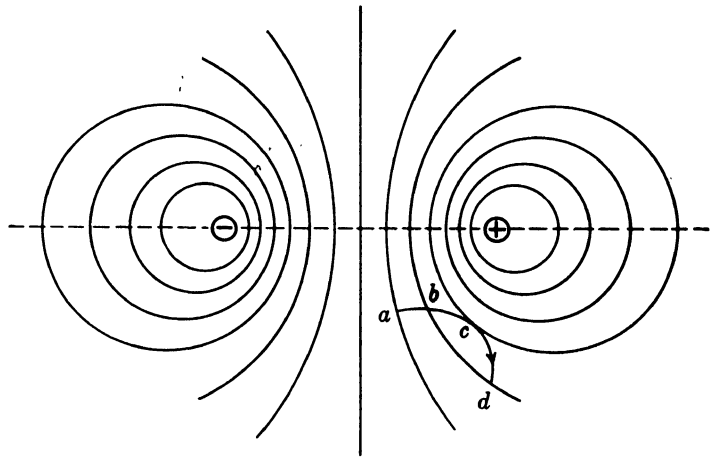


FIG. 8.4. Equipotential surfaces between two unlike charges.

charge of appreciable size is brought into this field, it will itself distort these equipotential surfaces.

When a voltage is connected across a resistor in series with a capacitor a current will start to flow in the circuit as though the capacitor is not in the circuit. Actually, electrons build up a charge on the capacitor plates that develop a voltage across the capacitor and the current in the circuit decreases exponentially from a maximum to a zero value. The current in this circuit is known as a *charging* or *displacement current* because it is the current required to displace electrons on the capacitor plates and develop a voltage across the capacitor plates. No electrons or current passes through the capacitor; hence, a continuous current can not exist in a perfect capacitor. However, if the capacitor has an imperfect dielectric a *continuous* current usually known as a *leakage current* will also exist.

8.7. Charging a Capacitor through a Resistor. If a current of i amperes flows for t seconds to charge a capacitor, the charge q in coulombs on the capacitor is

$$q = it \quad \text{coulombs} \tag{8.4}$$

Solving this equation for i gives

$$i = \frac{q}{t} \quad \text{amp} \tag{8.4a}$$

This equation gives the value of a steady charging current i for t seconds. Usually this is not the case. A more general statement of the charging current at any instant is given by the rate of change of charge with respect to time. If it is desired to know the current at some particular instant, it can be found by plotting a curve of the charge on the capacitor, as ordinate, against time, as abscissa. Such a curve is shown in Fig. 8.5; the charge q is plotted as a function of the time t. The current i at any

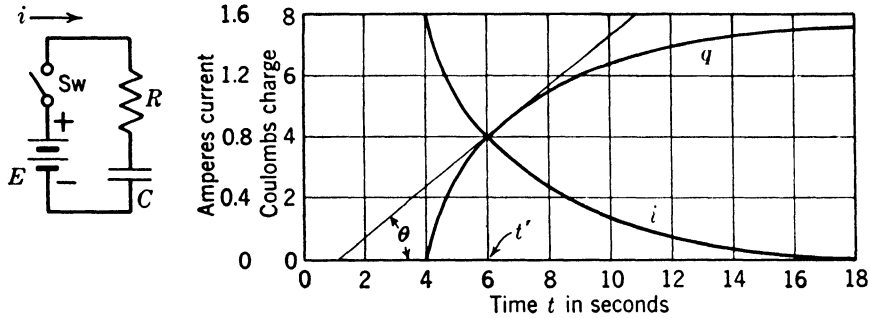


FIG. 8.5. Charge q and current i as a function of time t in a series circuit containing resistance and capacitance.

instant is equal to the slope of this curve. For instance, at t' it is desired to find the value of the current i. Draw a line tangent to the curve at this point, as shown in Fig. 8.5. The slope of this line is

$$\tan \theta = \frac{\text{opp}}{\text{adj}}$$

or

$$i = \tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{6 - 1} = 0.8 \text{ amp}$$

the charging current at time t'. A curve of the charging current i is plotted in Fig. 8.5, and at this instant it is 0.8 amp.

Generally, this type of problem is stated

$$i = \frac{dq}{dt} \tag{8.5}$$

which says that the current i is equal at every instant to the slope of the curve of the charge q plotted as a function of the time t. Instead of drawing a tangent and making a large triangle to find the current, an infinitesimally small triangle is drawn, which is made up of an opposite side dq and adjacent side dt, as shown in Fig. 8.6. In calculus, the charging current i could be called the *derivative* of the charge with respect to

the time, which means the slope of the curve shown in Fig. 8.6 and stated in Eq. (8.5). Calculus in this case is a convenient way of expressing the slope of the curve.

8.8. Dielectric Strength. When the electrostatic field intensity is increased to a certain point, the strain produced in the dielectric becomes too great and the substance "breaks down." At this point the electrons are forced out of their orbits, which means that a current flows through the dielectric. The *dielectric strength* is measured by the volts required to cause a unit thickness of the material to fail. The dielectric strength of a material depends upon its chemical nature and the thickness of the pieces of material. The dielectric strength of a material usually decreases as the temperature increases, because the electrons have more energy, owing to the heat, and consequently can jump their orbits with a lower field intensity. Since power loss in a dielectric is proportional to frequency, more heat is developed at high frequencies and the dielectric strength is decreased. This is an important point to remember in r-f work.

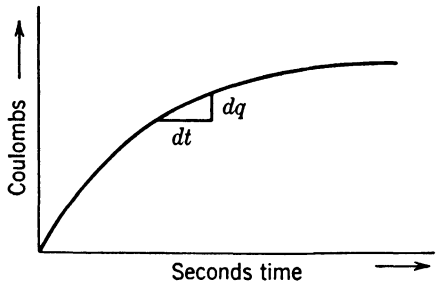


FIG. 8.6. Charge as a function of time.

8.9. Corona. When the electrostatic charge density on a conductor becomes too great, the dielectric breaks down. Since the electrostatic charge density is greater at sharp points on the surface of the conductor, the dielectric will break down first at these points. The breakdown known as the *corona effect* is manifest visually by an electrostatic glow or luminous discharge and audibly by a hissing sound.

Corona becomes manifest at a *critical voltage*, which depends upon atmospheric conditions when the dielectric is air. Corona losses increase very rapidly with voltage increases beyond the critical voltage, until the *disruptive critical voltage* is reached. At this point the voltage gradient (volts per inch) reaches a maximum, which for air is about 75,000 volts/in.

8.10. Electrostatic Capacity. When one volt applied across the terminals of a capacitor causes a charge of *one* coulomb on each side, the capacity is said to be one *farad*. Mathematically,

$$q = Ce \qquad \text{coulombs} \tag{8.6}$$

where q = instantaneous charge, coulombs
C = capacity, farads
e = instantaneous voltage, volts

Since the farad is such a large unit, the microfarad is used in ordinary work, and the micromicrofarad is used at radio frequencies.

8.11. Capacity of a Parallel-plate Capacitor. The capacity of a capacitor having parallel plates is

$$C = \frac{225KA}{d} \qquad \mu\mu f \tag{8.7}$$

where C = capacity, $\mu\mu f$
K = dielectric constant ($K = 1$ for air)
A = area of plates, sq in.
d = separation of the plates, mils ($\frac{1}{1000}$ in.)

This equation neglects the edge effects, because of the assumption that the plates are very close together.

When more than 2 plates are used

$$C = \frac{225KA(n-1)}{d} \qquad \mu\mu f \tag{8.8}$$

where n is the number of plates in the capacitor.

Example 8.1. Determine the capacity of a 23-plate air capacitor made of plates with an area of 2.25 sq in. and a spacing of 31.82 mils.

Solution. Using Eq. (8.8)

$$C = \frac{(225)(1)(2.25)(22)}{31.82}$$
$$= 350\ \mu\mu f \qquad \textit{Ans.}$$

8.12. Capacity of a Transmission Line. The capacity of two long parallel wires is given by the following equation:

$$C = \frac{0.01941K}{\log \frac{D-r}{r}} \tag{8.9}$$

where C = capacity, μf/mile
K = dielectric constant ($K = 1$ for air)
D = distance between conductor centers
r = radius of conductor

D and r must be in the same units. Since they form a ratio, it does not matter what units are used.

Example 8.2. Find the capacity of 55 miles of No. 4 copper wire spaced 18 in. apart.

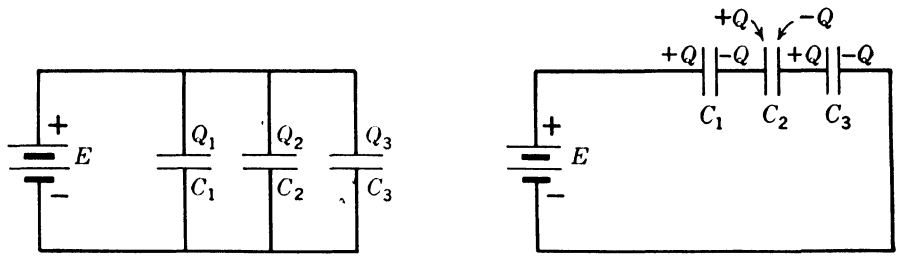
Solution. From wire tables we find that No. 4 wire has a radius of 0.10215 in.

$$C = \frac{(0.01941)(55)}{\log \frac{18 - 0.10215}{0.10215}} = \frac{(0.01941)(55)}{\log 175.3}$$
$$= \frac{1.068}{2.2438} = 0.4758\ \mu f \qquad \textit{Ans.}$$

8.13. Capacity of Capacitors in Parallel. When capacities are connected side by side, the resultant capacity is the sum of the individual capacities; for example,

$$C = C_1 + C_2 + C_3 \tag{8.10}$$

This is shown in Fig. 8.7*a* and is explained by the fact that the same potential is across each capacity, and the total charge is the sum of the



(a) Parallel connection (b) Series connection

FIG. 8.7. Capacitors in parallel (left) and in series (right).

charges on the various capacitors. The total charge Q can be written by use of Eq. (8.6) as follows:

$$Q = Q_1 + Q_2 + Q_3$$

where $Q_1 = C_1 e$
$Q_2 = C_2 e$
$Q_3 = C_3 e$

Hence,
$$Q = C_1 e + C_2 e + C_3 e$$
$$= (C_1 + C_2 + C_3)e = Ce$$

where $C = C_1 + C_2 + C_3$ as given in Eq. (8.10).

8.14. Capacity of Capacitors in Series. When a voltage E is applied across a series of capacitors, equal and opposite charges will be induced on each pair of plates connected together. The total voltage across the capacitors will then be the sum of the voltages across the various capacitors. Applying Eq. (8.6), the total voltage will be

$$e = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} + \frac{Q_3}{C_3}$$

But $Q = Q_1 = Q_2 = Q_3$, as explained here. Hence

$$e = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) Q$$

The equivalent capacity C is then

$$C = \frac{e}{Q} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

or

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \tag{8.11}$$

If more than three capacities are in series, they are added in the same way that the three were added in Eq. (8.11).

If only two capacities are connected in series,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

Or by solving for C, this can be written

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{8.12}$$

which says that the capacity of two capacitors connected in series is equal to the product of their capacities divided by their sum.

Example 8.3. Three capacitors, having capacities of 200, 500, and 1,000 $\mu\mu$f, are connected in series. What is the equivalent capacity?

Solution

$$\frac{1}{C} = \frac{1}{200} + \frac{1}{500} + \frac{1}{1{,}000}$$

Reducing to a common denominator

$$\frac{1}{C} = \frac{5 + 2 + 1}{1{,}000} = \frac{8}{1{,}000}$$

$C = 125$ $\mu\mu$f = the equivalent capacity *Ans.*

8.15. Energy in an Electrostatic and an Electromagnetic Field. The electrostatic energy stored in a capacitor is given by

$$W = \tfrac{1}{2}CV^2 \tag{8.13}$$

where W = energy, joules
C = capacity, farads
V = voltage, volts

This equation states that the energy W in a capacitor varies directly as the capacity C of the capacitor and the square of the voltage V across it.

Since $Q = CV$, Eq. (8.13) can be written

$$W = \tfrac{1}{2}QV \tag{8.14}$$

The electromagnetic energy stored in an inductance coil takes on a

similar form as follows:

$$W = \tfrac{1}{2}LI^2 \tag{8.15}$$

where W = energy, joules
L = inductance, henrys
I = current, amp

This equation states that the energy W stored in an inductance coil varies directly with the inductance L of the coil and the square of the current I flowing through it.

If a perfect capacitor C is charged with a voltage V, it will have an energy of W given by Eq. (8.13). Now, if this capacitor is connected across a perfect inductance L, the voltage across the capacitor will cause a current to flow in the inductance. As this process goes on, the energy of the electrostatic field is decreasing and the energy of the electromagnetic field is increasing, until the voltage across the capacitor becomes zero. At this instant, the current in the inductance has reached a maximum, and all the energy stored in the capacitor as electrostatic energy has been transferred to the magnetic field of the inductance coil in the form of electromagnetic energy. Equations (8.13) and (8.15) are equal in this case, so that

$$\tfrac{1}{2}LI^2 = \tfrac{1}{2}CV^2 \tag{8.16}$$

Solving this equation for I gives

$$I = V\sqrt{\frac{C}{L}} \tag{8.17}$$

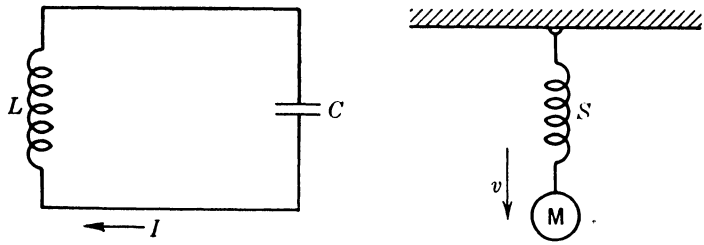
which shows that the current I that will flow in the inductance will be large if V and C are large. The current I will continue to flow even after the voltage of the capacitor has become zero, because of the inertia effect of the inductance. As it continues to flow, the capacitor will become charged in the opposite direction until the maximum voltage V is reached in the opposite direction. The current will then be zero again, and all the energy of the magnetic field will be in the electrostatic field. The voltage V across the capacitor can be determined from Eq. (8.16). It is

$$V = I\sqrt{\frac{L}{C}} \tag{8.18}$$

This equation shows that the voltage across the capacitor will be very high if the capacity C is low while the current I and inductance L are large. This may result in dangerously high values of voltage across the capacitor.

The preceding electrical circuit is shown in Fig. 8.8*a*, and its mechanical analogue in Fig. 8.8*b*. It is easy to see that if the weight is pulled

down, the spring is stretched, storing up potential energy, This is similar to the charged capacitor. When the weight is released, the force exerted by the spring will cause the weight to gain velocity, giving it kinetic energy. At some point, the force exerted by the spring will be zero, and its velocity, which corresponds to current in the other case, will be a maximum. All the potential energy that was in the spring due



(a) Electric system (b) Mechanical system

FIG. 8.8. Equivalent oscillating electric and mechanical systems.

to its stretched position is now in the moving weight as kinetic energy. So *in an oscillating mechanical system, the maximum potential energy is equal to the maximum kinetic energy,* while *in an oscillating electric system, the maximum electrostatic energy is equal to the maximum electromagnetic energy.*

8.16. Damping. In any practical electric circuit or mechanical system, there will be resistance or *damping,* as it is sometimes called. This will cause a part of the energy to be dissipated on each oscillation, and eventually the system will come to rest when all the energy has been dissipated in the resistance of the circuit. This dissipated energy will appear in the form of heat in either case.

This type of oscillation is known as *free* oscillation, because no energy is supplied to the system after it is set into oscillation. If energy is supplied, *forced* oscillations result.

Summary

Electrical capacity is analogous to a mechanical spring in that it stores and releases energy. The rate of charge and discharge of a capacitor varies exponentially with respect to time. When the capacitor is first connected in a circuit, a heavy charging current flows, with a resulting high drop across the resistance in the circuit. As the capacitor becomes charged, the current decreases, and more of the charging potential appears across the plates of the capacitor.

An electrostatic field is set up between the unlike charges on opposite plates of a capacitor. A force F will then act between the two bodies. Its magnitude can be determined as follows:

$$F = \frac{Q_1 Q_2}{K d^2}$$

where F = force acting between the two bodies, dynes
Q = charge on the two bodies, coulombs
d = distance of separation, cm
K = dielectric constant of the medium between the two plates

A vacuum has a *dielectric constant* of 1. Solid dielectrics having greater values will weaken the field between the plates and increase the capacity. The dielectric constant of air is practically equal to 1.

An *equipotential surface* lies at right angles to the electrostatic lines of force and is a surface along which a charge may be moved without doing work or having work done on it. The volt is the measure of the amount of work required to move a charge from one equipotential surface to another in the direction of the lines of force.

A *displacement current* is the changing of an electric field with respect to time and can take place in a vacuum, or a solid dielectric where the electrons are not free to move as in a conductor, but can be displaced in their orbits in the individual atoms.

The current in a capacitor is equal to the rate of change of charge with respect to time, or $i = dQ/dt$. When the charge on a capacitor is zero, then a large current flows, and when the capacitor is fully charged, the current is zero, with the possible exception of a small leakage current through the dielectric.

Dielectric strength is measured in the volts required to break down a unit thickness of the dielectric. Thus, for any dielectric, the breakdown voltage that will puncture it is a function of the chemical composition and thickness of the material. The unit of capacity is the *farad*, which is equal to the capacity of a body that has a potential of 1 volt when holding a charge on its plates of 1 coulomb.

$$C = \frac{Q}{E} \qquad \text{or} \qquad Q = CE$$

where Q = charge, coulombs
C = capacity, farads
E = potential difference, volts

As the farad is too large to work with in practice, the units *microfarad* (μf), one one millionth of a farad, and micromicrofarad ($\mu\mu$f), one million-millionth of a farad, are customarily used in radio and communications work.

The capacity of a parallel-plate capacitor may be determined from the following formula:

$$C = \frac{225KA}{d}$$

where C = capacity, $\mu\mu$f
K = dielectric constant
A = area of single plate, sq in.
d = separation of plates, mils $\frac{1}{1000}$

If n interleaved plates are used, this becomes

$$C = \frac{225KA(n-1)}{d}$$

The capacity between long parallel wires is

$$C = \frac{0.01941}{\log \frac{D-r}{r}}$$

where C = capacity, μf/mile
K = dielectric constant
D = distance between conductor centers
r = radius of conductor
D and r must be in the same units.

If capacitors C_1, C_2, and C_3 are connected in parallel, the total capacity C is

$$C = C_1 + C_2 + C_3$$

If they are connected in series, the total capacity is

$$C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}} \quad \text{or} \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

For two capacities in series, this is

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

The energy W stored in a capacitor of size C charged to voltage V is

$$W = \tfrac{1}{2}CV^2 \quad \text{joules}$$

When a capacitor and inductance are connected in parallel, energy will be transferred back and forth between the electrostatic field of the capacitor and the electromagnetic field of the inductor.

Then as

$$W = \frac{1}{2}LI^2 = \frac{1}{2}CV^2, \qquad I = V\sqrt{\frac{C}{L}}$$

As the current flows from inductance to capacitor and back again, electrical oscillations result. These oscillations would continue indefinitely were it not for the resistance of the circuit, which damps the oscillations until all the energy has been dissipated and the oscillations cease.

Exercises

8.1. What happens in a capacitor when it is charged? Explain.

8.2. State Coulomb's law in your own words.

8.3. Define and explain "electrostatic potential."

8.4. What will be the charge on a capacitor if 0.001 amp flows for 10 sec?

8.5. Explain your concept of a displacement current. What kind of currents flow in a leaky capacitor?

8.6. Under normal conditions, what voltage is necessary to break down a 6-in. air gap?

8.7. What causes corona? How does it manifest itself?

8.8. A twisted pair of No. 18 wires has a separation of $\frac{1}{8}$ in. between centers. The rubber covering has a dielectric constant of 7. What is the capacity between wires per mile? Number 18 wire has a diameter of 40.3 mils.

CHAPTER 9

SINUSOIDAL ALTERNATING CURRENTS

9.1. Simple Alternator. To obtain an alternating current or voltage, a coil of wire can be rotated at a constant angular velocity in a uniform magnetic field. For the sake of simplicity, consider one turn of wire, as shown in Fig. 9.1, rotating in a uniform magnetic field. According to the theory presented in Chap. 6, the voltage across the slip rings, through which the current is collected by means of brushes, can at any instant be written, by Eq. (6.1),

$$e = -\frac{d\Phi}{dt}10^{-8} \quad \text{volts} \qquad (9.1)$$

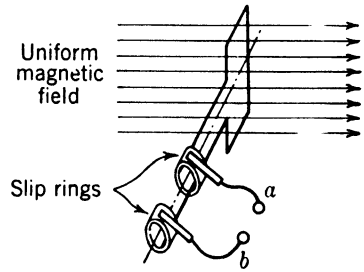


FIG. 9.1. Simple a-c generator.

where e is the instantaneous voltage in volts, and $d\Phi/dt$ is the rate of change of flux through the loop. It will be recalled that in Chap. 6 this equation dealt with a wire moving at right angles to a magnetic field. If this wire were not moved at the same velocity all the time, or if the flux did not have a uniform density, this voltage would vary. In such a case, the voltage will depend on the rate of cutting flux, and hence will be the tangent to a curve of flux linking the circuit plotted against time. Referring to Fig. 9.2, for position θ we have

$$\frac{d\Phi}{dt} = \frac{a}{b} \qquad (9.2)$$

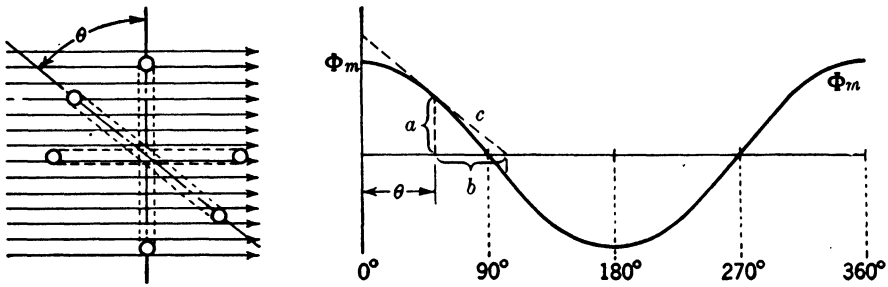
For N turns in the coil, Eq. (9.1) is written

$$e = -N\frac{d\Phi}{dt}10^{-8} \quad \text{volts} \qquad (9.3)$$

The negative sign is used because the force exerted on the wire by the magnetic field is opposite to the force causing the coil to turn.

In order to develop the equation for a sinusoidal voltage, consider the turn of wire in the magnetic field of Fig. 9.1. When the coil is in the vertical position, it will link the maximum amount of flux Φ_m. This is shown for $\theta = 0°$ in Fig. 9.2*b*. If the coil is rotated through an angle θ to a new position, as shown in Fig. 9.2*a*, the coil will link fewer lines of force, and this value is represented by the ordinate a in Fig. 9.2*b* with the

angle θ as the abscissa. When the coil is in a horizontal position, it will link no flux, and the ordinate will be zero, as indicated in Fig. 9.2*b* for 90°. To continue the rotation of the coil will cause flux to pass through the coil in the opposite direction. This is illustrated in Fig. 9.2*b* by drawing the curve below the axis (giving negative values). When the coil is again vertical, it will link Φ_m lines of force, represented by $\theta = 180°$ in the figure. Continuing the rotation, the flux will decrease to zero again when the coil is horizontal ($\theta = 270°$) and then link flux in the



(a) Coil in field (b) Instantaneous flux curve

FIG. 9.2. Diagram and graph for various positions of an armature coil in a uniform magnetic field, showing flux for various angles of rotation.

original direction as the rotation is continued to the original vertical position. The coil has turned completely around, or moved one *cycle*, and is ready for the next similar cycle.

The curve in Fig. 9.2*b* is a *cosine curve*, and mathematically the flux Φ can be written as a cosine function of θ; thus,

$$\therefore \Phi = \Phi_m \cos \theta \tag{9.4}$$

To check this, let $\theta = 0°$; then the loop of wire will be in a vertical position and will link Φ_m flux. Looking in a table, $\cos 0° = 1$, and the flux is then $\Phi = (\Phi_m)(1)$, which shows that the equation is correct for this one particular case. When $\theta = 90°$, $\cos 90° = 0$, and no flux is linked, which again checks the curve of Fig. 9.2*b*.

The next consideration is to obtain the slope of this curve at each point; this can be done by substituting Φ of Eq. (9.4) in Eq. (9.3), giving

$$e = -\frac{N\, d(\Phi_m \cos \theta)}{(10^8)\, dt} \tag{9.5}$$

For $\theta = 0°$, a line drawn tangent to the curve is horizontal and parallel to the x-axis. Hence the slope is zero. Take another point θ as shown in Fig. 9.2*b*, and draw a line tangent to the curve and extended to the x-axis forming the triangle with sides a and b. It is important to note

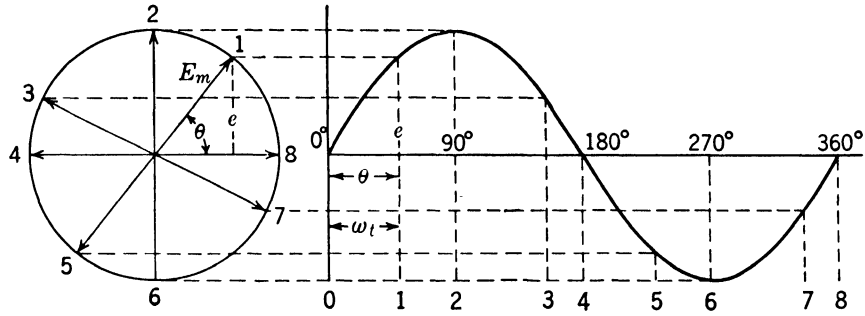
that the line slopes down when going to the right, making the slope negative. In Eq. (9.5)

$$d(\Phi_m \cos\theta) \propto a \qquad \text{while} \qquad dt \propto -b$$

or

$$\frac{d(\Phi_m \cos\theta)}{dt} = \frac{a}{-b} \tag{9.6}$$

The sign $\propto$ means *proportional to.* If calculus were used, an infinitesimally small triangle would be made at the point on the curve corresponding to θ, and this very small triangle would be exactly similar to the large triangle having sides a and b. Since the slope in each case is the same, Eq. (9.6) is justified. Consider $\theta = 90°$; the slope is a maximum



(a) Revolving vectors (b) Generation of a sine curve

FIG. 9.3. Graphical representation of a sine curve.

negative value at this point. That is, $a/-b$ has the largest negative value that will be reached. Going on to $\theta = 180°$, the slope becomes zero, and then for larger values of θ, it becomes positive, reaching a maximum positive slope at 270° and then decreasing gradually to zero again at 360°. A plot of the negative of this curve is shown in Fig. 9.3*b* and is known as a *sine curve.* The maximum instantaneous voltage e_m of this curve occurs when the loop of wire is in a horizontal position (coil links zero flux). In this position, one side is moving down across the magnetic field at right angles to it, and the other side of the loop is moving up at right angles to the field. In this position, the rate of change of flux linkage is a maximum. The magnitude of this maximum instantaneous voltage e_m will depend directly upon the angular velocity ω, the maximum flux linked by the coil, and the number of turns in the coil. With this information, Eq. (9.5) for one turn can be written

$$e = -\frac{\Phi_m \omega(-\sin\theta)}{10^8} \tag{9.7}$$

The negative sign in front of sin θ is due to the negative slope obtained from Fig. 9.2*b*. The angle θ can be written as an angular velocity ω

multiplied by the time t; thus

$$\theta = \omega t \tag{9.8}$$

With this substitution, Eq. (9.7) for θ can be written

$$e = \frac{\Phi_m \omega}{10^8} \sin \omega t \tag{9.9}$$

And for N turns in the coil

$$e = \frac{N\Phi_m \omega}{10^8} \sin \omega t \tag{9.10}$$

where e = instantaneous voltage, volts
N = number of turns in coil
Φ_m = maximum flux linkage of the coil
ω = angular velocity of the coil, radians/sec
t = time, sec

The maximum voltage developed in the coil can now be written as the coefficient of sin ωt; thus

$$e_m = \frac{N\Phi_m \omega}{10^8} \tag{9.11}$$

Making this substitution in Eq. (9.10) gives

$$e = E_m \sin \omega t \tag{9.12}$$

If this sinusoidal alternating voltage is placed across a pure resistance, an instantaneous current i will flow that is proportional to this voltage at all times according to Ohm's law, and hence can be written

$$i = \frac{E_m}{R} \sin \omega t = I_m \sin \omega t \tag{9.13}$$

Example 9.1. A 100-turn coil rotates at a uniform velocity of 377 radians per sec in a uniform magnetic field and has a maximum flux linkage of 10^6 lines of force. If the voltage is zero at time $t = 0$, what is the instantaneous voltage when $t = 0.001$ sec? If the resistance of the circuit is 100 ohms, write the equation of the instantaneous current.

Solution. Substituting in Eq. (9.10)

$$e = \frac{(100)(10^6)(377)}{10^8} \sin [(377)(0.001)] = 377 \sin (0.377)$$

$$= 377 \sin (21.6°) = (377)(0.368) = 138.7 \text{ volts} \qquad \textit{Ans.}$$

Substituting in Eq. (9.13)

$$i = \frac{377}{100} \sin (377t) = 3.77 \sin (377t) \text{ amp} \qquad \textit{Ans.}$$

9.2. Vectors Applied to Alternating Quantities. In order to be able to work a-c problems easily by Ohm's and Kirchhoff's laws, vectors are

introduced. In the author's book, "Applied Mathematics," Chap. 6, Vector Addition, space vectors only were considered. A *space vector* is used to represent quantities that are fixed in magnitude and direction. Such quantities as a steady force, a steady magnetic field, or a steady electrostatic field may be represented by a space vector.

To handle alternating quantities, *revolving vectors* are brought into play. A revolving vector has a constant magnitude and revolves about one end at a constant angular velocity. Sometimes it is called a *time vector* because it changes position with changes in time. Figure 9.3*a* illustrates eight positions of a revolving vector. The length of the ordinate from the free end of the vector to the x-axis represents the instantaneous values. These ordinates are marked in Fig. 9.3*b* for the eight positions of the revolving vector shown in Fig. 9.3*a*. The vector revolves counterclockwise.

Take, for example, position 1 of Fig. 9.3*a*. The instantaneous voltage e is found from the following equation:

$$e = E_m \sin \theta$$

where E_m = the hypotenuse of the triangle
e = the opposite side of the triangle
θ = the angle E_m makes with the x-axis

Since $\theta = \omega t$, this equation is the same as Eq. (9.12), which is true for any position of E_m.

It is very easy to plot a sine curve by projecting the free end of the vector on the y-axis. These projections represent the instantaneous values of voltage. If the angle θ is marked off along the x-axis, these projections can be extended until they intersect the proper ordinate, which will locate a point on the curve. For instance, position 1 of Fig. 9.3*a* is projected to position 1 of Fig. 9.3*b*. The dotted line shows this projection. Similarly, position 2 is projected from Fig. 9.3*a* to Fig. 9.3*b*, and so on, until the complete curve has been determined. To obtain a smooth sine curve, it is usually desirable to plot more than 8 points; 12 or 16 points would be preferable.

If the curve is plotted for a vector starting to rotate from a vertical position, as shown in Fig. 9.4, instead of from a horizontal position as shown in Fig. 9.3, the result will be a cosine curve instead of a sine curve. It will be noted that e is measured from the positive y-axis (vertical line) in Fig. 9.4*a* for a cosine curve, while it was measured from the positive x-axis (horizontal line) in Fig. 9.3*a* for a sine curve. The equation for the cosine curve is

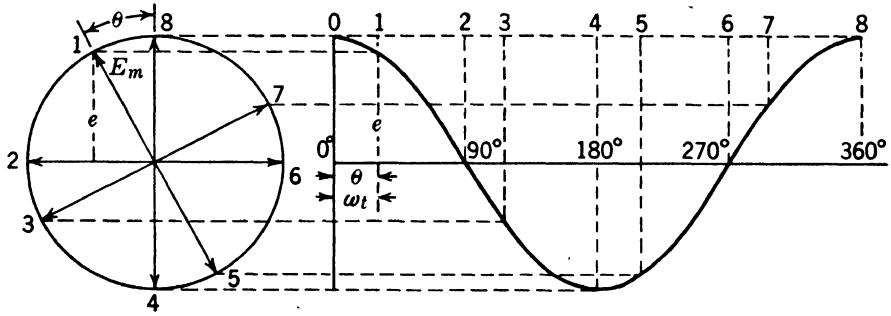
$$e = E_m \cos \theta = E_m \cos \omega t \tag{9.14}$$

where e = instantaneous voltage, volts
E_m = maximum voltage, volts
ω = angular velocity, radians/sec
t = time, sec
$\theta = \omega t$ = the angle E_m makes with the reference axis (y-axis)

The only difference between Figs. 9.3 and 9.4 is that the curve is shifted 90°. Equation (9.12) can be made into Eq. (9.14), if 90° is added to ωt; thus

$$e = E_m \sin (\omega t + 90) = E_m \cos \omega t \tag{9.15}$$

Usually the positive x-axis is used as a reference line. Then, to write the equation for a cosine curve, 90° must be added to start the curve when the vector E_m is in a vertical position.



(a) Revolving vector (b) Generation of a cosine curve

FIG. 9.4. Graphical representation of a cosine curve.

Equation (9.15) is the result of the following mathematical identity:

$$\cos \omega t = \sin (\omega t + 90°) \tag{9.16}$$

For instance, let $t = 0$ in this equation; then the cosine of 0° is 1, and the sine of 0 + 90 is also 1. If other values of ωt are substituted in this equation, it will be found to hold true for all values. In general, the angle will not be 90°, hence can be represented by some other angle ϕ; thus

$$e = E_m \sin (\omega t + \phi) \tag{9.17}$$

where ϕ is the phase angle. The phase angle ϕ is used, which means that the vector is displaced by an angle ϕ counterclockwise from the reference line (positive x-axis).

9.3. Addition of Alternating Quantities. Consider the sum of two alternating voltages, as shown in Fig. 9.5, where in general

$$e_1 = E_{m1} \sin (\omega t + \phi_1) \tag{9.18}$$

and

$$e_2 = E_{m2} \sin (\omega t + \phi_2) \tag{9.19}$$

The total sum can be written

$$e_t = e_1 + e_2 = E_{m1} \sin (\omega t + \phi_1) + E_{m2} \sin (\omega t + \phi_2) \quad (9.20)$$

or

$$e_t = E_{mt} \sin (\omega t + \phi_t) \quad (9.21)$$

where E_{mt} = the vector sum of E_{m1} and E_{m2}
= the total maximum instantaneous voltage

and ϕ_t = the phase angle E_{mt} makes with the reference axis, as shown in Fig. 9.5*a*

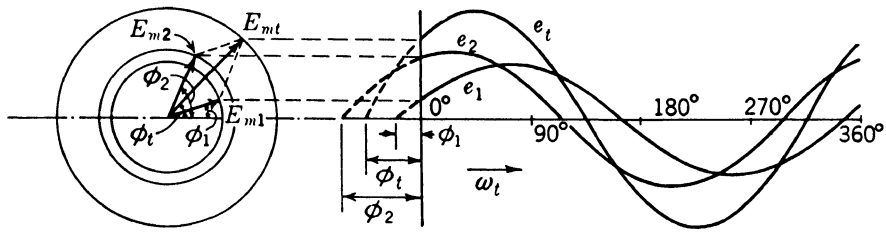


FIG. 9.5. Addition of sinusoidal vectors.

By applying vector addition (see Chap. 6 of "Applied Mathematics"), the magnitude of the total maximum voltage is found to be

$$E_{mt} = \sqrt{(E_{m1} \cos \phi_1 + E_{m2} \cos \phi_2)^2 + (E_{m1} \sin \phi_1 + E_{m2} \sin \phi_2)^2} \quad (9.22)$$

and the phase angle it makes with the reference axis can be obtained by solving the following equation for ϕ_t:

$$\tan \phi_t = \frac{E_{m1} \sin \phi_1 + E_{m2} \sin \phi_2}{E_{m1} \cos \phi_1 + E_{m2} \cos \phi_2} \quad (9.23)$$

Figure 9.5 shows how instantaneous voltages e_1, e_2, and e_t, which are continuously changing with respect to each other, can be represented by the vectors E_{m1}, E_{m2}, and E_{mt}, which are stationary with respect to each other. In other words, E_{m1}, E_{m2}, and E_{mt} revolve together in a counterclockwise direction at an angular velocity ω, as spokes in a wheel revolve around their axle. The solution of a-c problems is materially simplified by using these revolving vectors. These vectors are perfectly general in their nature; hence they can represent voltage, current, resistance, reactance, and impedance vectors. In fact, any revolving vector can be treated by this analysis. When adding vectors representing voltage or current, care should always be taken that the voltages and currents are of the same frequency. If the frequencies are the same, the vectors maintain the same relationship with each other throughout the cycle. If the frequencies are different, the vector diagram must be drawn for a given instant of time.

Example 9.2. Two 100-volt (maximum) generators are connected in series. If the maximum voltage of the second generator is 90° ahead of that of the first one, determine the instantaneous voltage of their sum with reference to the first-generator voltage.

Solution. By Eq. (9.18) the first-generator voltage can be written

$$e_1 = 100 \sin (\omega t + 0°) = 100 \sin \omega t$$

Similarly, by Eq. (9.19) the second-generator voltage can be written

$$e_2 = 100 \sin (\omega t + 90°) = 100 \cos \omega t$$

By Eq. (9.20)

$$e_t = e_1 + e_2 = 100 \sin \omega t + 100 \cos \omega t$$

But by Eq. (9.22)

$$E_{mt} = \sqrt{[(100)(1) + (100)(0)]^2 + [(100)(0) + (100)(1)]^2}$$
$$= \sqrt{100^2 + 100^2} = 141.4$$

And by Eq. (9.23)

$$\tan \phi_t = \frac{(100)(0) + (100)(1)}{(100)(1) + (100)(0)} = \frac{1}{1} = 1$$

or

$$\phi_t = 45°$$

And the resulting total instantaneous voltage by Eq. (9.21) is

$$e_t = 141.4 \sin (\omega t + 45°) \qquad \textit{Ans.}$$

9.4. Average and Effective Values of Voltage and Current. The *average value* of *voltage* or *current* in an a-c circuit is taken as the average value of the ordinate over a half cycle. Obviously, the average value of an entire cycle of alternating voltage or current equals zero, which is the reason that the average is taken over a half cycle in this definition. The average ordinate can be determined by dividing the area above the x-axis under the curve by the length of the half cycle. This is shown in Fig. 9.6 and can be written in the form of an equation as follows:

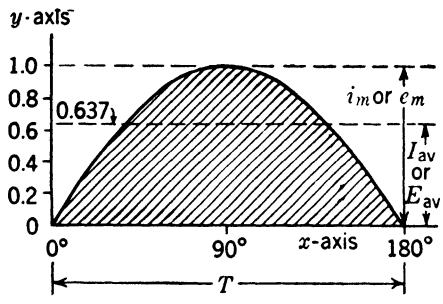


FIG. 9.6. Graph of half-cycle showing average ordinate.

$$I_{av} = \frac{\text{area under a half cycle of instantaneous currents}}{T} \qquad (9.24)$$

$$E_{av} = \frac{\text{area under a half cycle of instantaneous voltages}}{T} \qquad (9.25)$$

where T is the length of the half cycle as shown in Fig. 9.6. If i_m or e_m, the maximum value, is unity, as shown in Fig. 9.6, I_{av} or E_{av} will have a

value of 0.637 when the wave is a sine wave. In general, this can be written

$$I_{av} = 0.637 i_m \tag{9.26}$$

where I_{av} = average current over a half cycle of instantaneous currents
and i_m = maximum instantaneous current

Likewise

$$E_{av} = 0.637 e_m \tag{9.27}$$

where E_{av} = average voltage over a half cycle of instantaneous voltages
e_m = maximum instantaneous voltage

The most important fact to know about an electric circuit, however, is not the average current or voltage passing through the circuit. A knowledge of the average values of these quantities may give a very distorted picture of what is taking place. The characteristic of a circuit that gives us much more information is the *power* developed in the circuit. The power developed by current flowing through a resistor is given at any instant by

$$p = i^2 R$$

If the current is sinusoidal the instantaneous value is expressed by

$$i = I_m \sin 2\pi f t$$

where I_m is the maximum value of the current during 1 cycle, then the instantaneous value of the power dissipated in a resistance will be given by

$$p = R I_m^2 \sin^2 2\pi f t \tag{9.28}$$

and the total energy supplied to the resistor in the interval of time $t = t_2 - t_1$ is

$$W = \int_{t_1}^{t_2} p \, dt \tag{9.29}$$

The average power, P is the average rate at which energy is supplied over the prescribed period of time $t = t_2 - t_1$. The average power is determined by dividing the total energy supplied in the period of time t by the time. In equation form this is expressed

$$P = \frac{W}{t} = \frac{1}{t} \int_{t_1}^{t_2} p \, dt = \frac{1}{T} \int_0^T p \, dt \tag{9.30}$$

where $t = T$, the period of time from time t_1 to time t_2. Applying this equation to determine the average power dissipated in the resistor substitute Eq. (9.28) in Eq. (9.30) and solve, thus,

$$P = \frac{1}{T}\int_0^T RI_m^2 \sin^2 2\pi ft\, dt$$
$$= \frac{RI_m^2}{2\pi fT}\int_0^T \sin^2 2\pi ft\, d(2\pi ft)$$
$$= \frac{RI_m^2}{2\pi fT}\left[\frac{2\pi fT}{2} - \frac{\sin 4\pi ft}{4}\right]_0^T$$
$$= \frac{RI_m^2}{2\pi fT}\left[\frac{2\pi fT}{2} - \frac{\sin 4\pi fT}{4}\right]$$
$$= \frac{RI_m^2}{2} \qquad (9.31)$$

because as shown in Fig 9.6 the period of time T is for one complete power cycle and therefore the sine term must equal zero.

The power in an a-c circuit is defined as equal to average power P, and the value of the direct current that would produce the same average power is called the *effective* value of the current I. The power that would be developed by a current I in a resistor R will, of course, be given by,

$$P = I^2R \qquad (9.32)$$

where P = average power, watts
I = effective current, amp
R = effective resistance, ohms

This equation is true in an a-c circuit as well as in a d-c circuit. However, in an a-c circuit the current will be continuously varying. The current will sometimes be considerably higher than the effective value, other times zero, and every half cycle it will change its direction. The effective value as determined here takes account of these variations, and since heating is not a function of the direction of current flow, the average power can be expressed in terms of the effective current as given or in terms of the effective voltage as developed below.

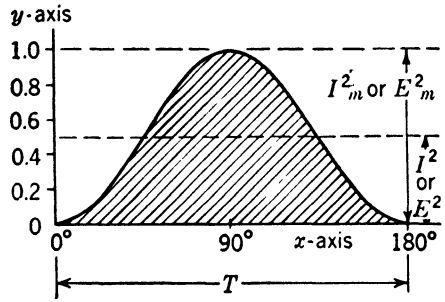


FIG. 9.7. Graph of half-cycle showing squared values as ordinates.

From Ohm's law $I = E/R$ or $I^2 = E^2/R^2$ which, if substituted in Eq. (9.32), gives

$$P = \frac{E^2}{R} \qquad (9.33)$$

where P = average power, watts
E = effective voltage, volts
R = effective resistance, ohms

Comparing Eq. (9.31) and Eq. (9.32) it is apparent that $I_m^2/2 = I^2$ as shown in Fig. 9.7. Taking the square root of both sides of this equation results in

$$I = \frac{I_m}{\sqrt{2}} = 0.707 I_m \tag{9.33a}$$

A similar derivation for the voltage gives

$$E = \frac{E_m}{\sqrt{2}} = 0.707 E_m \tag{9.34}$$

The ratio of the maximum value to the effective value is known as the *crest* or *peak factor*. For a sine wave it is

$$\frac{I_m}{I} = \frac{E_m}{E} = \frac{1}{0.707} = 1.414 \tag{9.35}$$

It is interesting to note that the square root of 2 gives the same value, $\sqrt{2} = 1.414$. This factor becomes important in cases where the wave is not sinusoidal. For instance, in neon-sign lighting it is desirable to have a very high crest factor (which means that there are high voltage maximums) that will ionize the gas.

The *form factor* is defined as the ratio of the effective value to the average value and for a sine wave is

$$\frac{I}{I_{av}} = \frac{E}{E_{av}} = \frac{0.707}{0.637} = 1.11 \tag{9.36}$$

It will be noted that no subscripts are used with effective values. This is because the effective values are most common. Practically all measuring instruments are calibrated in effective values.

The effective value of current or voltage can be determined from the rms (which means root-mean-square) value of the wave. The rms value of a sine wave is 0.707 times the maximum value, which of course coincides with the effective value. If the wave is nonsinusoidal then its rms value can be determined by taking the square root of the sum of the squares of the current or voltage values over one complete period.

Example 9.3. If the average a-c voltage developed across a 200-ohm resistance is 90 volts, determine the average power consumed by the resistance.

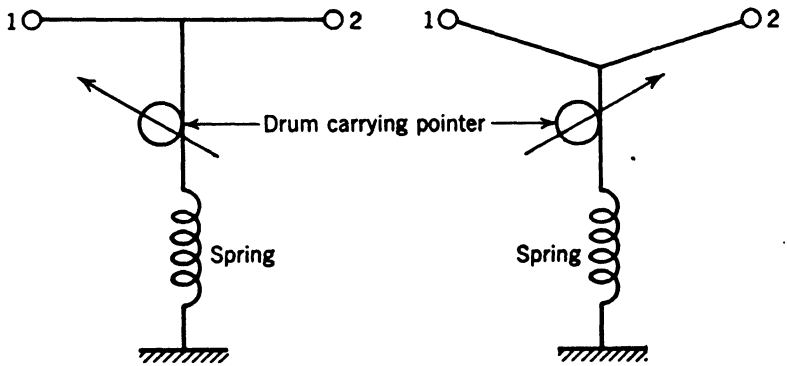
Solution. By Eq. (9.33) the effective voltage is

$$E = 1.11 E_{av} = (1.11)(90) = 100$$

By Eq. (9.35) the average power is

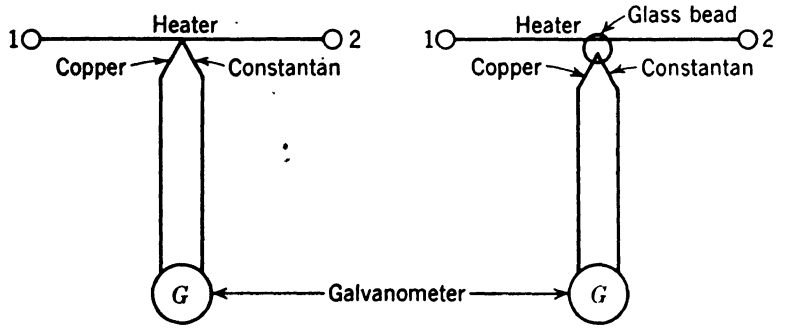
$$P = \frac{E^2}{R} = \frac{100^2}{200} = 50 \text{ watts} \qquad \textit{Ans.}$$

9.5. High-frequency A-c Meters. Two types of high-frequency a-c meters are in use. Both meters depend upon a heating effect to measure the current flowing. One meter is known as a *hot-wire ammeter* and is shown in Fig. 9.8. The other type is the *thermocouple ammeter* and is shown in Fig. 9.9.



(a) Zero current (b) Maximum current

FIG. 9.8. Hot-wire ammeter.



(a) Contact type (b) Heater type

FIG. 9.9. Thermocouple meters.

9.6. Hot-wire Ammeter. The hot-wire ammeter depends upon the expansion of a heated wire when current is passed through the wire. Figure 9.8*a* illustrates the position of the wire between terminals 1 and 2 of the meter when no current is flowing. Now if a current is passed through the wire between terminals 1 and 2, the wire will become elongated. The spring will keep the wire taut and in doing so will rotate the small drum carrying the pointer, as shown in Fig. 9.8*b*.

The stretching of the wire depends upon the *heating effect of the current*,

which means that the movement is *proportional to the square of the current.* If the scale is marked uniformly, the meter is known as a *current-squared meter.*

The hot-wire ammeter has been replaced in most uses by the thermocouple meter. The defects of the hot wire ammeter are

1. Large temperature error
2. Lack of perfect resilience in the heated expansion wire
3. Low overload capacity
4. Unsuitability for low-current ranges
5. Too large inductance for high frequencies

9.7. Thermocouple Meter. The thermocouple meter, which does not have these defects, depends for its operation upon the fact that a voltage will be generated if two dissimilar metals are in contact and heated, as shown in Fig. 9.9*a*. Usually copper and constantan are used, or if a very sensitive meter is desired, bismuth and antimony can be used as the two dissimilar metals. The voltage generated will be a d-c voltage; hence it does not depend upon the direction of current flow in the heater element. A d-c microammeter or galvanometer can be used to detect the current flow. As in the hot-wire ammeter, the deflection of the thermocouple meter is proportional to the heating effect, and hence proportional to the square of the effective current being measured. Some thermoelements are placed in a vacuum to increase their speed and sensitivity and reduce the temperature errors. At the higher frequencies, objectionable coupling between the circuits is removed, at the expense of speed and sensitivity, by heating the thermal junction through a small bead of glass as illustrated in Fig. 9.9*b*.

Example 9.4. A current-squared galvanometer thermocouple meter is graduated in 100 evenly spaced divisions. The full-scale reading of the meter is 120 ma. Determine the current for a scale reading of 83 divisions. What will the scale reading be if the current is reduced to 75 per cent of the 83-division current?

Solution. Using proportion and the fact that the scale reading is proportional to the square of the current

$$\frac{100}{120^2} = \frac{\text{scale divisions}}{\text{current squared}} = \frac{\text{scale divisions}}{I^2}$$

For this example

$$I^2 = \frac{(120)^2(83)}{100} = 11{,}952$$

Hence the current for a scale reading of 83 divisions is

$$I = \sqrt{11{,}952} = 109.3 \text{ ma} \qquad \textit{Ans.}$$

If the current is reduced to 75 per cent of this value, the new current will be

$$I' = (0.75)(109.3) = 82 \text{ ma},$$

and the corresponding scale reading will be

$$\text{Scale divisions} = \frac{(100)I'^2}{120^2} = \frac{(100)(82)^2}{120^2} = 46.7 \qquad \textit{Ans.}$$

9.8. Audio-frequency Contact Rectifier Meter. The most efficient low-current a-c meter is a contact rectifier used in conjunction with a d-c meter. The most common type employs a copper oxide disk rectifier. In order to increase the sensitivity, commercial instruments contain a full-wave rectifier made of four copper oxide rectifiers connected in the form of a bridge, as shown in Fig. 9.10. This type of meter has less accuracy than the thermocouple meter but has the advantage of not being sluggish. Owing to the large input capacity, this type of meter is limited to the a-f range. The principal disadvantage, however, is its inaccuracy due to variation of rectifying properties with time and temperature. The full-scale accuracy is limited to about 5 per cent as compared to 2 per cent, which is the usual accuracy of thermocouple meters.

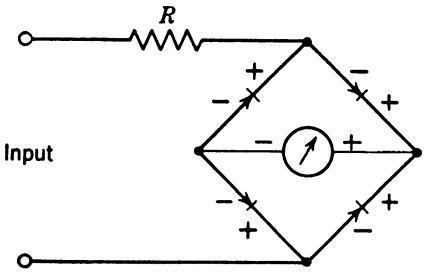


FIG. 9.10. Bridge circuit for a copper oxide meter.

The contact rectifier has a high resistance and hence is a good voltmeter. Voltmeters with a full-scale range of 0.5 volt and a resistance of 2,500 ohms are common. This type of meter is widely used as a db or vu meter across program or audio circuits.

Summary

The voltage e induced in a coil of wire rotating in a uniform magnetic field is directly proportional to the number of turns in the coil and to the time rate at which the number of flux lines linking the coil is changing. As an equation, this is

$$e = -N\frac{d\Phi}{dt}10^{-8}$$

The relationship between the maximum flux Φ_m and the flux Φ linking the coil at any angle θ is given by the expression

$$\Phi = \Phi_m \cos\theta$$

It is now necessary to find the rate of change of $\Phi_m \cos\theta$ with respect to time, and from differential calculus the differential of the cosine equals the sine. In other words, the rate of change of a cosine curve at any instant is equal to the sine of the angle at that instant. Thus, a sine wave is displaced 90° from a cosine curve, as in Fig. 9.11. When curve a is zero, its rate of change is maximum, and the sine curve b is 1. When the curve is maximum, its instantaneous rate of change is zero.

The induced voltage is then proportional to the angular velocity at which the coil is rotating (this gives time rate of change) multiplied by the sine of the angle at any instant. This voltage is then

$$e = \frac{N\Phi_m\omega}{(10^8)} (\sin \omega t)$$

where e = instantaneous voltage, volts
N = number of turns
Φ_m = maximum flux linkage of coil
t = time from zero or starting point, sec

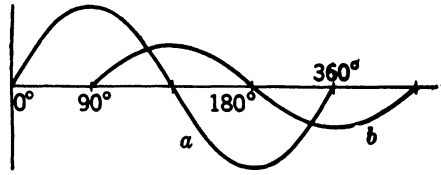


FIG. 9.11. Sine curve a and cosine curve b.

The maximum voltage E_m is then

$$E_m = \frac{N\Phi_m\omega}{10^8}$$

and the voltage at any instant is

$$e = E_m \sin \omega t$$

If this voltage is impressed across a resistance R, then the current I at any instant is

$$i = I_m \sin \omega t$$

The instantaneous voltage may be represented by a vector of length proportional to the maximum voltage and rotating around the origin at the same angular velocity ω. Then the voltage at any instant is the projection of the vector on the y-axis, or

$$e = E_m \sin \theta$$

If two or more voltages are impressed on the same circuit, with their vectors rotating at the same velocity ω but having different phase angles so that

$$e_1 = E_{m1} \sin (\omega t + \phi_1)$$
$$e_2 = E_{m2} \sin (\omega t + \phi_2)$$

then the total maximum voltage when $t = 0$ is

$$E_{mt} = \sqrt{(E_{m1} \cos \phi_1 + E_{m2} \cos \phi_2)^2 + (E_{m1} \sin \phi_1 + E_{m2} \sin \phi_2)^2}$$

and the total voltage in the circuit at any instant is

$$e_t = E_{mt} \sin (\omega t + \phi_t)$$

where $\tan \phi_t = \dfrac{E_{m1} \sin \phi_1 + E_{m2} \sin \phi_2}{E_{m2} \cos \phi_1 + E_{m2} \cos \phi_2}$

Average value of an alternating current or voltage is equal to the area under the half-cycle curve of instantaneous values divided by the length of the half cycle. For sinusoidal variation this is

0.637 (maximum value)

Effective value of voltage or current is equal to the square root of the average of the squares of the instantaneous values over a half cycle. For sine or cosine wave, it is 0.707 times the maximum height of the wave, or for current and voltage

$$E = 0.707E_m; \qquad I = 0.707I_m$$

As heat or power is proportional to the square of the current, the effective figure

is equivalent to the same value of direct current and is the most widely used value of an alternating current or voltage.

High-frequency a-c meters operate on the heating effect of current. The *hot-wire ammeter* has a wire that expands when heated by a current and the elongation is converted into the angular rotation of a needle. This type is relatively inaccurate and is not widely used.

The *thermocouple meter* measures the current generated at the junction of two dissimilar metals when they are heated. This direct current is read on a d-c milliammeter whose scale is calibrated in terms of the current through the thermocouple necessary to produce that current.

These meters have a scale marked off according to the current squared, as the heat increases by the square of the current in the wire.

For audio frequency purposes, a copper oxide rectifier can be used to rectify the alternating current and a d-c movement reads the value of rectified current.

Exercises

9.1. Explain how the flux in a magnetic field must be cut to produce a sinusoidal voltage.

9.2. What is a phase angle? How is it used?

9.3. (*a*) Plot a cosine curve of instantaneous values of flux Φ over a complete cycle, when the maximum flux linking the circuit is 1.1×10^{10} lines. (*b*) What flux links the circuit at 40°, 70°, 270°, and 350°?

9.4. (*a*) If the angular velocity is $\omega = 377$ and there are 50 turns of wire used in the field of Exercise 9.3, determine the maximum voltage that will be generated in the coil. (*b*) What will be the effective voltage generated? (*c*) Write the equation of the instantaneous voltage. (*d*) What will be the instantaneous voltage when time $t = 2$ sec?

9.5. Define: crest factor, peak factor, effective value, average value, maximum value, and form factor.

9.6. What will be the maximum current in an a-c circuit that will produce the same heating effect as 4 amp in a d-c circuit? Give your reasons. What is the average value of alternating current flowing?

9.7. If a circuit has 15 ohms effective resistance and the maximum sinusoidal voltage across the resistance is 81 volts, what is the power in watts dissipated in the resistance?

9.8. If a squared-current meter registers 2.7 scale divisions and the current is tripled, what will the meter register?

9.9. If

$$e_1 = 120 \sin \omega t$$
$$e_2 = 150 \cos \omega t$$

plot each curve and the curve of their sum e_t, over a complete cycle. What is the phase angle of e_t?

CHAPTER 10

EFFECTIVE RESISTANCE, INDUCTIVE AND CAPACITIVE REACTANCE

In Chaps. 6 and 8 of this book, a study was made of self-inductance and electrostatic capacity. The effect of opening and closing a switch in a circuit containing a constant voltage was treated for the case of a pure resistance, a resistance in series with an inductance, and a resistance in series with a capacity. These cases are inadequate for a thorough treatment of a-c (alternating-current) circuit theory; hence a more general treatment of alternating voltages applied to circuits containing resistance, inductance, and capacity will be given in this chapter.

In Chap. 6 it was shown that a conductor carrying current is surrounded by an electromagnetic field. It will be recalled that each electron is a negative charge with electrostatic lines of force terminating on it. If this electron is set in motion, these electrostatic lines of force have to move along with the electron, and as this occurs, a magnetic field is created. In other words, a moving electron represents a current, and a current always has an electromagnetic field associated with it. Therefore, we can roughly say that *an electromagnetic field is the result of an electrostatic field in motion.*

As was pointed out in Chap. 8, the total energy in the electromagnetic and electrostatic fields of an oscillator consisting of a pure inductance connected in series with a pure capacity is constant but sinusoidally changing from one field to the other. Also, the current through the coil and the voltage across the capacitor vary sinusoidally. Since there is no source for supplying energy in this system, an a-c generator, which can supply energy, will be used in this chapter. However, in the case of a pure inductance or a pure capacity load, the a-c generator will merely supply energy sinusoidally for the electromagnetic or electrostatic field. For one half cycle it will put energy into the field, and for the next half cycle it will remove all this energy from the field. Energy can be absorbed from the generator only by a resistance load.

In this chapter Ohm's law will be extended to cover the voltage-current relationship existing in a-c circuits.

10.1. Effective Resistance. When an alternating voltage is applied to a circuit, several effects are noticed that are different from the effects produced by applying a d-c (direct-current) voltage to the circuit.

Since the current will be continuously varying, the electromagnetic field will be continuously expanding and collapsing. If the electromagnetic field links other circuits or conducting material, secondary currents will be established, resulting in *eddy-current losses*, and if the materials are magnetic in nature, *hysteresis losses* will also be present. Also, the current density will not be uniform over the cross section of wire, owing to *skin effect*. For one or more of these reasons the effective resistance of a conductor to alternating current is greater than its ohmic resistance to direct current. The *effective* or apparent *resistance* of a circuit is determined by the amount of power that will be dissipated when an alternating current is passed through the circuit. The square of the effective current multiplied by the effective resistance gives the true average rate at which heat energy is dissipated.

The *effective resistance* of a circuit can be defined by the equation

$$R = \frac{P}{I^2} = \frac{E^2}{P} \tag{10.1}$$

where R = effective resistance, ohms
P = average dissipated power, watts
I = effective current, amp
E = effective voltage, volts

This equation says that the effective resistance of an a-c circuit can be found by measuring the total average power dissipated as heat divided by the square of the effective current flowing, or the square of the effective voltage divided by the total average dissipated power. The resistance R in all a-c formulas is to be understood as *effective resistance* as defined above.

10.2. Alternating-current and Voltage Wave Form in a Pure Resistance. Using the equations of instantaneous voltage and current developed in Chap. 9, we have, from Eqs. (9.12) and (9.13),

$$e = E_m \sin \omega t \tag{10.2}$$
$$i = I_m \sin \omega t \tag{10.3}$$

According to Ohm's law, the instantaneous voltage e will produce an instantaneous current i through a pure effective resistance R; thus

$$e = Ri \tag{10.4}$$

Substituting for i,

$$e = RI_m \sin \omega t$$

Substituting for e,

$$E_m \sin \omega t = RI_m \sin \omega t$$

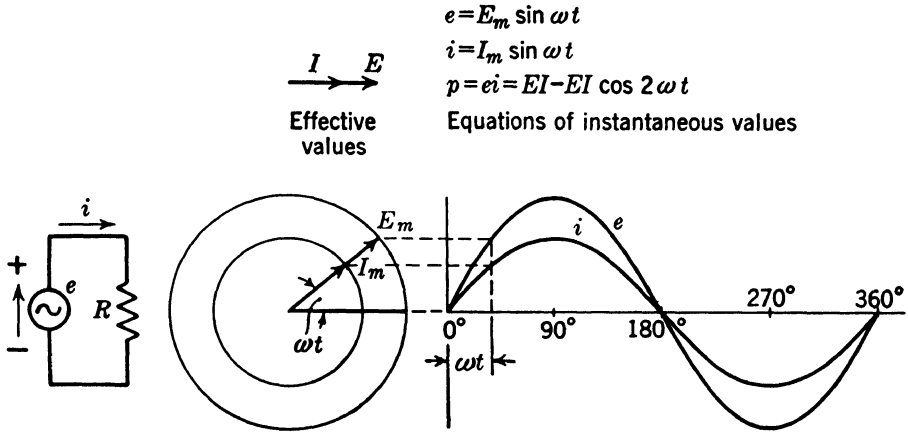
Dividing both sides by sin ωt, this equation becomes

$$E_m = RI_m \tag{10.5}$$

This equation says that the maximum instantaneous voltage is equal to the effective resistance multiplied by the maximum instantaneous current. This equation of Ohm's law written in effective values is

$$E = RI \tag{10.6}$$

The current is said to be *in phase* with the voltage because they pass through zero and maximum values at the same time. Consider an a-c generator connected to a pure resistance as shown in Fig. 10.1*a*. When



(a) Circuit (b) Maximum values (c) Graph of instantaneous values

FIG. 10.1. Alternating-current circuit containing a pure resistance.

the instantaneous voltage e is in the direction of the arrow (pointing up from the negative to the positive), an instantaneous current i will flow clockwise around the circuit, as indicated, through the resistance R. Since the voltage determines the magnitude of the current (resistance fixed), it is convenient to represent the current and voltage vectors (see the vector diagram of E_m and I_m in Fig. 10.1*b*). Usually the effective values are represented in a vector diagram as shown above in this figure. *The current and voltage being in phase means that they lie along the same straight line.* If the maximum vector values are rotated, they will trace out the instantaneous values of current and voltages as shown in Fig. 10.1*c*. The equations for these curves are given in Fig. 10.1*c*.

Example 10.1. A load resistance carries 5 amp with 10 volts of direct current applied, and only 4 amp with 10 volts of alternating current applied. Determine (*a*) the resistance of the load in each case, (*b*) the average power dissipated by the load in each case.

Solution. (*a*) By Ohm's law the d-c ohmic resistance is

$$R = \frac{E}{I} = \frac{10}{5} = 2 \text{ ohms} \qquad \textit{Ans.}$$

Similarly, the a-c effective resistance is

$$R = \frac{E}{I} = \frac{10}{4} = 2.5 \text{ ohms} \qquad \textit{Ans.}$$

(*b*) By solving Eq. (10.1), the average d-c power consumed by the resistance load is found to be

$$P = I^2R = (5)^2(2) = 50 \text{ watts} \qquad \textit{Ans.}$$

or

$$P = \frac{E^2}{R} = \frac{10^2}{2} = 50 \text{ watts} \qquad \textit{Check}$$

Similarly, the average a-c power consumed by the resistance load is

$$P = I^2R = (4^2)(2.5) = 40 \text{ watts} \qquad \textit{Ans.}$$

or

$$P = \frac{E^2}{R} = \frac{10^2}{2.5} = 40 \text{ watts} \qquad \textit{Check}$$

10.3. Alternating-current and Voltage Wave Form in a Pure Inductance. The rate at which current changes in an inductance determines the voltage induced across the inductance. To derive this relationship, we have, from Eq. (6.1)

$$e = -N \frac{d\Phi}{dt} 10^{-8} \text{ volts} \tag{10.7}$$

the instantaneous voltage across a coil of N turns when the rate of change of flux linkage in the coil is $d\Phi/dt$.

From the definition of inductance as given in Eq. (7.6)

$$L = N \frac{d\Phi}{di} 10^{-8} \text{ henrys} \tag{10.8}$$

Multiplying both sides of Eq. (10.7) by dt gives

$$e\,dt = -N\,d\Phi 10^{-8}$$

and multiplying both sides of Eq. (10.9) by di gives

$$-L\,di = -N\,d\Phi 10^{-8}$$

Since things equal to the same thing are equal to each other

$$e\,dt = -L\,di$$

or

$$e = -L \frac{di}{dt} \tag{10.9}$$

This equation states that the instantaneous voltage e induced across an inductance L can be determined by multiplying the slope of the curve of instantaneous current plotted against time by the inductance of the circuit. To illustrate this equation, consider an alternator generating a sinusoidal voltage connected to a pure inductance, as shown in Fig.

10.2*a*. Now, referring to Fig. 10.2*c*, it will be noted that the slope of the instantaneous current curve i is a maximum at 0°. Then the instantaneous voltage e must be a maximum at this instant, as shown in the figure. When the current i reaches a maximum, the slope is zero, and hence di/dt is zero, and e from Eq. (10.9) must be zero. This occurs at 90° and is shown to be true in the figure. Continuing this reasoning process, any point on the instantaneous-voltage curve can be determined from the slope of the instantaneous-current curve and the value of inductance L in the circuit.

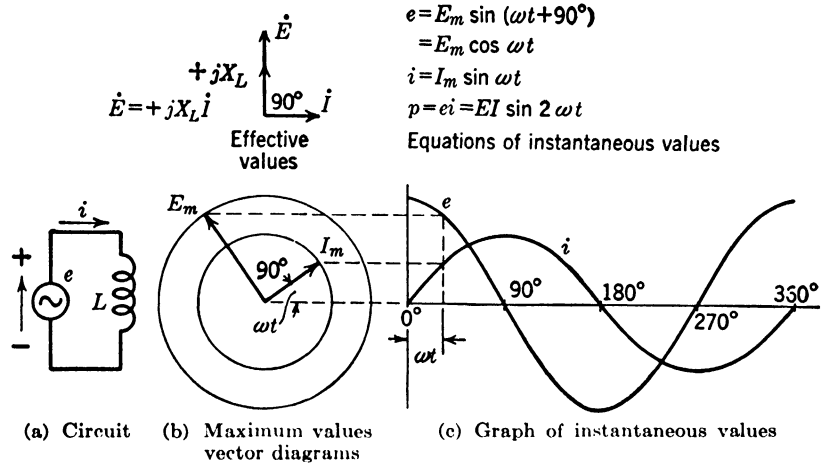


FIG. 10.2. Alternating-current circuit containing a pure inductance.

In order to trace out the instantaneous values of Fig. 10.2*c*, it is convenient to use rotating vectors E_m and I_m as shown in Fig. 10.2*b*. These vectors rotate in a counterclockwise direction; hence, the current is said to *lag* the voltage in a pure inductive circuit by 90°. Usually the effective values are represented in a vector diagram as shown in Fig. 10.2*b*. The equation of the instantaneous values of voltage and current are given in Fig. 10.2*c*. The instantaneous currents form a sine curve, while the instantaneous voltages form a cosine curve.

Example 10.2. If the current in a 10-henry inductance changes, at a particular instant, at the rate of 50 amp/sec, what is the instantaneous voltage induced across the inductance at that instant?

Solution. By Eq. (10.9)

$$e = -L\frac{di}{dt} = -(10)(50) = -500 \text{ volts} \qquad \textit{Ans.}$$

10.4. Derivation of Inductive Reactance. In the circuit of Fig. 10.2*a* the voltage drops around the circuit can be written,

$$e - L\frac{di}{dt} = 0 \qquad (10.10)$$

This expression should not be confused with Eq. (10.9) which expresses the induced voltage in the inductance L. If the induced voltage was expressed e_L then Eq. (10.10) would read

$$e + e_L = 0$$

Now let us derive an expression for the inductive reactance by substituting for the instantaneous values in Eq. (10.10). These values are given in Fig. 10.2*c*, thus

$$E_m \cos \omega t = L \frac{d}{dt} (I_m \sin \omega t)$$

Performing the differentiation

$$E_m \cos \omega t = LI_m\omega \cos \omega t$$

Therefore

$$E_m = I_m\omega L$$

Converting from maximum to effective values and replacing ωL by $2\pi fL$ gives

$$E = 2\pi fLI \tag{10.11}$$

where E = effective voltage, volts
2π = 6.2832 radians in a circle of 360°
f = frequency, cycles/sec
L = inductance, henrys
I = effective current, amp

A second look at this equation reveals that it is very much like Ohm's law, especially if we define inductive reactance as

$$X_L = 2\pi fL \tag{10.12}$$

With this substitution Eq. (10.11) becomes

$$E = X_LI \tag{10.13}$$

where E = effective voltage, volts
X_L = inductive reactance, ohms
I = effective current, amp

Equation (10.12) reveals that the inductive reactance is directly proportional to the frequency and the inductance of the circuit. This means that the ohms of inductive reactance will increase as the frequency or inductance increases. Equation (10.12) is many times expressed

$$X_L = \omega L \tag{10.14}$$

where $\omega = 2\pi f$ is the angular velocity in radians/sec.

A circle contains 2π radians, so if the frequency f is 1 cycle per sec, the angular velocity will be 2π radians per sec, or the rotating vector will

complete 1 revolution every second. If the frequency $f = 60$, then the revolving vector will complete 60 revolutions every second or 377 radians per sec.

Equation (10.13) is true for magnitude only. If the phase relationship is taken into account, the operator j must be placed in front of X_L; thus, in vector notation

$$\dot{E} = jX_L\dot{I} \quad \text{(vector equation)} \qquad (10.15)$$

Usually the dots over the vectors E and I are omitted when it is understood that all quantities in the analysis are vector quantities. Referring to the vector diagram of effective values in Fig. 10.2*b*, Eq. (10.15) written in polar form is

$$\underline{E/90°} = (\underline{X_L/90°})(\underline{I/0°}) \qquad (10.16)$$

This equation says that the effective-current vector I along the positive x-axis multiplied by the inductive reactance X_L along the positive y-axis gives the effective-voltage vector E along the positive y-axis. The product of the magnitudes of X_L and I gives the magnitude of E, and the sum of their phase angles gives the phase angle of E.

Sometimes $\underline{/I}$ is written along the left-hand margin, indicating that the phase angle of all vectors are taken with reference to I, which has a zero phase angle. Thus, when it is understood that this is a vector equation with I as the reference vector, it is written simply

$\underline{/I}$ $$E = jX_LI \qquad (10.17)$$

Example 10.3. In a 2-henry choke coil there is an alternating current of 200 ma that has an angular velocity of 628.3 radians/sec. Determine (*a*) the reactance of the choke coil, (*b*) the voltage developed across the choke coil, and (*c*) the frequency of the current.

Solution. (*a*) By Eq. (10.14)

$$X_L = \omega L = (628.3)(2) = 1{,}256.6 \text{ ohms} \qquad \textit{Ans.}$$

(*b*) By Eq. (10.13)

$$E = X_LI = (1{,}256.6)(0.2) = 251.3 \text{ volts} \qquad \textit{Ans.}$$

(*c*) By the definition of ω following Eq. (10.14)

$$f = \frac{\omega}{2\pi} = \frac{628.3}{6.283} = 100 \text{ cycles/sec} \qquad \textit{Ans.}$$

10.5. Alternating-current and Voltage Wave Form in a Pure Capacity. The rate at which the voltage changes across a capacitor determines the charging current. Using the equation developed in Chap. 8, the instantaneous current is by Eq. (8.5)

$$i = \frac{dq}{dt} \qquad (10.18)$$

But from Eq. (8.6), the charge is equal to the capacity multiplied by the voltage; thus

$$q = Ce \tag{10.19}$$

Substituting this value of q in Eq. (10.18) gives

$$i = \frac{d(Ce)}{dt}$$

Since C is a constant, it can be factored out from the derivative, giving the same answer; hence

$$i = C\frac{de}{dt} \tag{10.20}$$

where i = instantaneous charging current, amp
C = capacity, farads
$\frac{de}{dt}$ = change in voltage across the capacitor in time dt

It will be noted that this equation is very similar to Eq. (10.10). The instantaneous voltage and current have been interchanged, and the

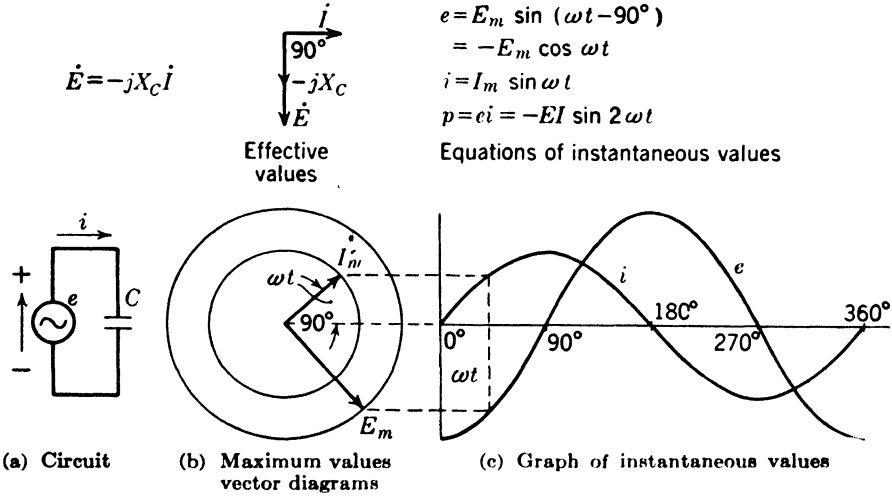


FIG. 10.3. Alternating-current circuit containing a pure capacity.

capacity C replaces the inductance L. In this equation the instantaneous charging current i flowing into the capacitor is determined by multiplying the capacity of the capacitor by the slope of the curve of instantaneous voltage e plotted against time t. This is shown in Fig. 10.3. The alternator, which generates a sinusoidal voltage, is connected to a capacity C as shown in Fig. 10.3*a*. Now, referring to Fig. 10.3*c*, it will be noted that the slope of the instantaneous-voltage curve is zero at 0°;

hence the instantaneous current i must be zero at this instant, as shown in the figure. When the instantaneous voltage C reaches the x-axis at 90°, its slope is a maximum, which, multiplied by the capacity C in the circuit, will give a maximum instantaneous current i. This reasoning process can be carried on to more points, and the result will be as shown in Fig. 10.3*c*.

In order to trace out these curves, consider the revolving vectors E_m and I_m shown in Fig. 10.3*b*. When I_m is along the positive x-axis, E_m will lie along the negative y-axis, or 90° behind I_m. Usually it is said that *the current leads the voltage in a pure capacitive circuit by* 90°. Ordinarily, effective values are used in vector diagrams, in order that the results will also be in effective values. Such a diagram is shown in Fig. 10.3*b*. It does not matter how the vectors are placed on the paper in solving a problem, but they must be drawn with the correct angle between them. In the above case, this angle is 90°. Sometimes it is desirable to indicate the reference vector along the left-hand margin as illustrated in Eq. (10.17).

The curves traced in Figs. 10.2*c* and 10.3*c* are correct after the alternator has been running a few cycles. After the switch connecting the alternator to the inductance or capacity is closed, the electromagnetic field must be established in one case, and the electrostatic field must be established in the other case, before a steady sine wave results. It will, in general, take a few cycles before the current and voltage are as shown in these figures. When the switch is first closed, a *transient current or voltage* is established, and when it dies away after a few cycles, a *steady-state* condition will be reached. The above diagrams show only the steady-state condition. The figures in Chaps. 6 and 8 show transients.

Example 10.4. If a 1.0-μf capacitor has a charging current of 2 ma, at a particular instant, what is the rate at which the instantaneous voltage across the capacitor is increasing?

Solution. By Eq. (10.20)

$$\frac{de}{dt} = \frac{i}{C} = \frac{0.002}{1.0 \times 10^{-6}}$$
$$= 2{,}000 \text{ volts/sec} \qquad \textit{Ans.}$$

10.6. Derivation of Capacitive Reactance. The capacitive reactance can be derived by substituting for the instantaneous values in Eq. (10.20) and performing the differentiation; thus

$$i = C\frac{de}{dt}$$
$$I_m \sin \omega t = C\frac{d}{dt}(-E_m \cos \omega t)$$
$$= CE_m\omega \sin \omega t$$

or
$$I_m = E_m\omega C$$

Converting maximum to effective values and replacing ωC by $2\pi fC$ gives

$$I = 2\pi fCE \tag{10.21}$$

where I = effective current, amp
2π = 6.2832 radians in a circle of 360°
f = frequency, cycles/sec
C = capacity, farads
E = effective voltage, volts

This equation can be made to look like the Ohm's law equation if the capacitive reactance is defined as

$$X_C = \frac{1}{2\pi fC} \tag{10.22}$$

Substituting this value in Eq. (10.21) gives

$$E = X_C I \tag{10.23}$$

where E = effective voltage, volts
X_C = capacitive reactance, ohms
I = effective current, amp

Equation (10.22) indicates that the capacitive reactance varies inversely with the frequency and the capacity of the circuit. This means that the ohms of capacitive reactance will decrease as the frequency or capacity is increased. This is opposite to the effect of an inductive reactance. Equation (10.22) may be written in another convenient form

$$X_C = \frac{1}{\omega C} \tag{10.24}$$

where $\omega = 2\pi f$ is the angular velocity in radians per second.

Equation (10.23) is true for magnitude only. If the phase relationship is taken into account, the operator $-j$ must be placed in front of X_C; thus in vector notation

$$\dot{E} = -jX_C\dot{I} \quad \text{(vector equation)} \tag{10.25}$$

Referring to Fig. 10.3*b*, the vector diagram of effective values written in polar form is

$$E\angle -90° = X_C\angle -90° I\angle 0° \tag{10.26}$$

This equation shows that the effective-current vector along the positive x-axis multiplied by the capacitive-reactance vector along the negative y-axis gives the effective-voltage vector along the negative y-axis.

With it understood that this is a vector equation, it can be written with the current vector I as a reference, in the following form:

$$E = -jX_C I \tag{10.27}$$

Example 10.5. A 0.003-μf capacitor is connected to a 100-volt generator, which has a frequency of 1 mc. Determine (*a*) the capacitive reactance, (*b*) the effective current, and (*c*) the angular velocity.

Solution. (*a*) By Eq. (10.22)

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(6.2832)(10)^6(0.003 \times 10^{-6})}$$
$$= 53.05 \text{ ohms-capacitive reactance} \qquad \textit{Ans.}$$

(*b*) By Eq. (10.21)

$$I = 2\pi fCE = (6.2832)(10)^6 \times (0.003 \times 10^{-6})(100)$$
$$= 1.885 \text{ amp-effective current} \qquad \textit{Ans.}$$

(*c*) By the definition of ω following Eq. (10.24)

$$\omega = 2\pi f = (6.2832)(10)^6 \text{ radians/sec} \qquad \textit{Ans.}$$

Summary

If alternating current is applied to a circuit, power is dissipated by the effective resistance of the circuit. This resistance loss is due to *eddy-current losses*, *hysteresis losses*, and the higher resistance of the conductor to alternating current due to *skin effects*. If these losses are lumped together, the apparent resistance R that they present can be given by

$$R = \frac{P}{I^2}$$

where R = the effective resistance
P = the power dissipated
I = the effective current

The alternating current and voltage in a pure resistance are always *in phase* and the dissipated power P is equal to I^2R or E^2/R.

In a pure inductance, the induced voltage is proportional to the rate of change of flux produced by a changing current in the inductance. When the rate of change of current is zero, the voltage is zero, and when the rate of change of current is maximum, the voltage is maximum. However, the value of the current is zero when its rate of change is maximum, and the current is maximum when its rate of change is zero. Therefore, the voltage and current in the inductance are 90° out of phase, with the current lagging the voltage.

The relationship between instantaneous voltage e and the maximum voltage E_m is

$$e = E_m \sin (\omega t + 90°) = E_m \cos \omega t$$

and the instantaneous current i and maximum current I_m are related by

$$i = I_m \sin \omega t$$

The term $2\pi fL$ is represented by X_L in the equation $E = X_L I$ and called the *inductive reactance* of the circuit. It is similar to the resistance term in Ohm's law, but instead of $R = E/I$, now $X_L = E/I$, where $X_L = \omega L$. As this reactance causes the current to lag the applied voltage by 90°, the equation $E = jX_L I$, written in complex notation, indicates this phase shift.

The voltage across the plates of a capacitor is proportional to the charge stored in the capacitor multiplied by the charging current. A phenomenon like that in the inductance occurs. When the current is maximum, the charge stored in the capacitor

is zero, and the voltage is zero; when the current is zero, the stored charge is maximum and therefore the voltage is maximum. The current at any instant is equal to the rate of change with respect to time or $i = dq/dt$. As $q = Ce$, then $i = C\,de/dt$, and

$$e = C\int i\,dt.$$

In a pure capacitance where this effect is present, the current leads the voltage by 90°. The mathematical expressions for voltage and current are

$$e = E_m \sin(\omega t - 90°) = E_m \cos \omega t$$

and

$$i = I_m \sin \omega t$$

The ratio of E over I is equal to $1/2\pi fC = 1/\omega C = X_C$, the capacitive reactance. To show the 90° phase difference, this is written in complex notation as

$$E = -jX_C I$$

Exercises

10.1. Describe your concept of an electrostatic field and an electromagnetic field. How are they related?

10.2. How is the vector diagram drawn for a circuit containing a pure resistance? If $I = 10$ amp and $E = 100$ volts, what is the effective resistance? Draw a vector diagram.

10.3. How is the vector diagram drawn for a circuit containing a pure inductance? If $I = 75$ ma and $E = 350$ volts in a circuit containing a pure inductance, what is its reactance? Draw a vector diagram.

10.4. How is the vector diagram drawn for a circuit containing a pure capacity? If $I = 20$ ma and $E = 90$ volts in a circuit containing a pure capacity, what is its resistance? Draw a vector diagram.

10.5. (*a*) Write the equation of L in terms of X_L and f. (*b*) Write the equation of C in terms of X_C and f.

10.6. (*a*) If the angular velocity is 5,000 radians/sec to make $X_L = 250$ ohms, what is the inductance? (*b*) If the capacity is 4.5 μf and the reactance is 900 ohms, what is the angular velocity? What is the frequency?

10.7. If an instantaneous voltage $e = 18$ volts at an instant when $di/dt = 140$ amp/sec what is the inductance value?

10.8. If the instantaneous charging current $i = 9$ ma when $de/dt = 77$ volts/sec., what is the capacity value?

10.9. (*a*) $L = 39$ henrys; $f = 60$ cycles. Find X_L. (*b*) $L = 18$ μh; $X_L = 1{,}400$ ohms. Find f. (*c*) $X_L = 20{,}000$ ohms; $f = 100$ kc. Find L.

10.10. (*a*) $C = 39$ $\mu\mu$f; $f = 20$ Mc. Find X_C. (*b*) $C = 1.80$ μf; $X_C = 700$ ohms. Find f. (*c*) $f = 1{,}370$ kc; $X_C = 290$ ohms. Find C.

CHAPTER 11

ALTERNATING-CURRENT IMPEDANCE AND POWER

11.1. Alternating-current Power in a Pure Resistance. Before taking various combinations of the elements, consider the a-c power in pure elements. First, consider a pure resistance as illustrated in Fig. 11.1.

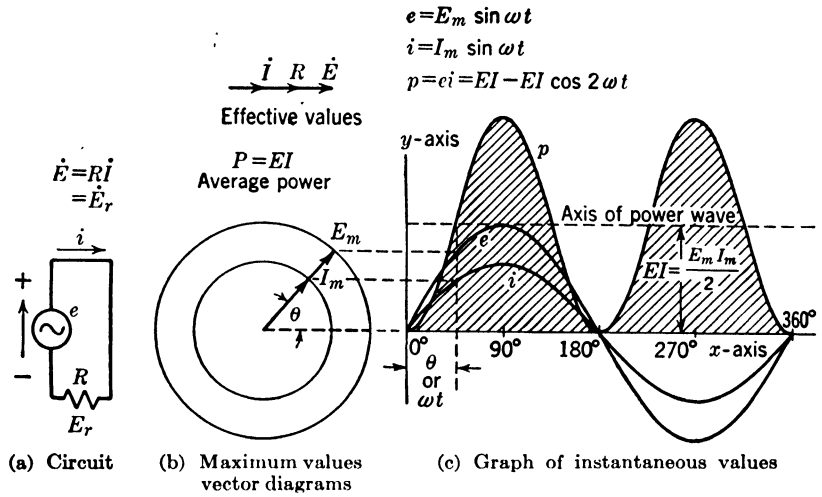


FIG. 11.1. Alternating-current power in a pure resistance.

The instantaneous power is always equal to the product of the corresponding instantaneous values of voltage and current. Mathematically this can be written

$$p = ei \tag{11.1}$$

where p = instantaneous power, watts
e = instantaneous voltage, volts
i = instantaneous current, amp

For a sine-wave alternating voltage and current, we can repeat the equations developed in Chap. 9 [see Eqs. (9.12) and (9.13)]

$$e = E_m \sin \omega t \tag{11.2}$$
$$i = I_m \sin \omega t \tag{11.3}$$

Substituting these sine-wave values in Eq. (11.1) gives

$$p = E_m I_m \sin^2 \omega t \tag{11.4}$$

From trigonometric identities, which can be found in "Applied Mathematics"[1]

$$\sin^2 \omega t = \frac{1 - \cos 2\omega t}{2} \tag{11.5}$$

Substituting this identity in Eq. (11.4) yields

$$p = \frac{E_m I_m}{2} - \frac{E_m I_m}{2} \cos 2\omega t \tag{11.6}$$

where p = instantaneous power, watts
E_m = maximum instantaneous voltage, volts
I_m = maximum instantaneous current, amp
$\omega = 2\pi f$ = the angular velocity, radians/sec
f = frequency, cycles/sec
t = time, sec

Referring to Fig. 11.1*c*, we see that the first term of Eq. (11.6) is $E_m I_m/2$, the distance from the x-axis up to the axis of the power wave. The second term in Eq. (11.6) is the equation of the power wave about its own axis (negative cosine curve). It should be noted in Fig. 11.1*c* that the power wave goes through 2 complete cycles while the voltage and the current waves complete only 1 cycle. The 2 in $\cos 2\omega t$ of Eq. (11.6) is responsible for this double-frequency effect of the power wave.

Equation (11.1) is perfectly general for any instantaneous-voltage and -current wave form. Equations (11.4) and (11.6) are equivalent equations for sinusoidal voltage and current in a pure resistance, but it is easier to analyze Eq. (11.6) to see just what is happening.

Example 11.1. In accordance with the preceding analysis, determine the instantaneous power in terms of maximum values when $\omega t = 0°$.

Solution. Solving Eq. (11.1) for $\omega t = 0°$,

$$p = ei = (0)(0) = 0 \qquad \textit{Ans.}$$

Solving Eq. (11.4) for $\omega t = 0°$,

$$p = E_m I_m \sin^2 \omega t = E_m I_m (0)^2 = 0 \qquad \textit{Check}$$

Solving Eq. (11.6) for $\omega t = 0°$,

$$p = \frac{E_m I_m}{2} - \frac{E_m I_m}{2} \cos 2\omega t$$

$$= \frac{E_m I_m}{2} - \frac{E_m I_m}{2} (1) = 0 \qquad \textit{Check}$$

In the solution of Eq. (11.6) in Example 11.1 the first term is the distance from the x-axis up to the power-wave axis, and the second term

[1]Carl E. Smith, *Applied Mathematics*, McGraw-Hill, 1945, pp. 305–306.

gives points with respect to the power-wave axis. To illustrate, at time $t = 0$ in Eq. (11.6) the first term is the distance from the x-axis up to the power-wave axis and the second term is the distance from the power-wave axis back down to the x-axis and hence the result is zero.

Example 11.2. Solve Example 11.1 when $\omega t = 90°$.
Solution. Solving Eq. (11.1) for $\omega t = 90°$,

$$p = ei = E_m I_m \qquad \textit{Ans.}$$

which is the maximum instantaneous power.
Substituting $\omega t = 90°$ in Eq. (11.4) yields

$$p = E_m I_m \sin^2 (90°) = E_m I_m (1)^2 = E_m I_m \qquad \textit{Check}$$

Substituting $\omega t = 90°$ in Eq. (11.6) also yields

$$p = \frac{E_m I_m}{2} - \frac{E_m I_m}{2} \cos (2)(90°) = E_m I_m \qquad \textit{Check}$$

In the solution of Eq. (11.6) in Example 11.2 the first term gives the distance from the x-axis up to the axis of the power wave, and the second term is the distance on up to the point of maximum instantaneous power. An inspection of Fig. 11.1*c* will reveal this graphically.

Example 11.3. Solve Example 11.1 when $\omega t = 270°$.
Solution. Solving Eq. (11.1) for $\omega t = 270°$,

$$p = ei = (-E_m)(-I_m) = E_m I_m \qquad \textit{Ans.}$$

Substituting $\omega t = 270°$ in Eq. (11.4) gives

$$p = E_m I_m \sin^2 (270°) = E_m I_m (-1)^2 = E_m I_m \qquad \textit{Check}$$

Substituting $\omega t = 270°$ in Eq. (11.6) gives

$$p = \frac{E_m I_m}{2} - \frac{E_m I_m}{2} \cos (2)(270°)$$

But $\cos (2)(270°) = \cos 540° = \cos (360 + 180) = \cos 180° = -1$
Note: Trigonometric functions repeat their values every 360°.
Substituting this result in the above equation gives

$$p = \frac{E_m I_m}{2} - \frac{E_m I_m}{2} (-1) = E_m I_m \qquad \textit{Check}$$

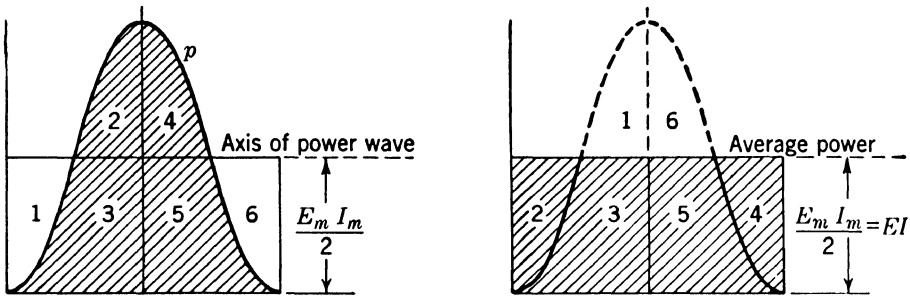
The average dissipated power is the area under the curve of instantaneous power p divided by the abscissa. This gives the average ordinate, which equals the average dissipated power.

The area under the instantaneous-power curve is shaded in Fig. 11.1*c*. For instance, take the abscissa from $\omega t = 0°$ to $\omega t = 180°$, which is a complete cycle for p. This portion of the curve of instantaneous power is redrawn in Fig. 11.2. The shaded area above the axis of the power

wave will exactly fill the area below this line; hence the average dissipated power in a pure resistance load is

$$P = \frac{E_m I_m}{2} \quad \text{(for a pure resistance)} \qquad (11.7)$$

where P = average dissipated power, watts
I_m = maximum instantaneous current, amp
E_m = maximum instantaneous voltage, volts
This is the distance from the x-axis up to the axis of the power wave. Comparing with Eq. (11.6), it is seen that the average dissipated power is



(a) Instantaneous power wave over a power cycle

(b) Average of instantaneous power wave over a cycle

FIG. 11.2. Averaging the instantaneous power in a resistive circuit.

equal to the first term. The second term averages zero over a complete cycle.

Figure 11.2*a* shows the actual distribution of instantaneous power over a full cycle of power or a half cycle of voltage or current. The shaded area marked 2 will exactly fill the vacant area marked 1 if the section is rotated 180°. Similarly the shaded area marked 4 will exactly fill the vacant area marked 6 if it is rotated 180°. Figure 11.2*b* shows the result of having all the shaded area below the axis of the power wave. The average ordinate equals the average dissipated power as given in Eq. (11.7).

Since effective values of voltage and current are more common, the following substitutions can be made in Eq. (11.7):

$$E_m = 1.414E = \sqrt{2}\,E \qquad (11.8)$$

$$I_m = 1.414I = \sqrt{2}\,I \qquad (11.9)$$

giving

$$P = EI \quad \text{(for a pure resistance)} \qquad (11.10)$$

where P = average dissipated power, watts
E = effective voltage, volts
I = effective current, amp

Example 11.4. A 60-cycle alternating current of 6 amp flows in a 200-ohm load resistor. Determine the average dissipated power, the maximum instantaneous power, and the frequency of the instantaneous-power wave.

Solution. By Ohm's law

$$E = IR = (6)(200) = 1{,}200 \text{ volts}$$

By Eq. (11.10) the average dissipated power is

$$P = EI = (1{,}200)(6) = 7.2 \text{ kw} \qquad \textit{Ans.}$$

The maximum instantaneous power is

$$p_m = E_m I_m = \sqrt{2}\, E\, \sqrt{2}\, I = 2EI$$
$$= 2(7.2) = 14.4 \text{ kw} \qquad \textit{Ans.}$$

Owing to the factor 2 in cos $2\omega t$, the frequency of the power wave is

$$2f = (2)(60) = 120 \text{ cycles/sec} \qquad \textit{Ans.}$$

11.2. Alternating-current Power in a Pure Reactance. In a pure inductance the current lags the voltage by 90°. From Fig. 10.2 we can write

$$e = E_m \cos \omega t \tag{11.11}$$
$$i = I_m \sin \omega t \tag{11.3}$$

Substituting these values in Eq. (11.1) gives

$$p = ei = E_m I_m \sin \omega t \cos \omega t \tag{11.12}$$

Substituting the trigonometric identity

$$\sin \omega t \cos \omega t = \frac{\sin 2\omega t}{2} \tag{11.13}$$

in Eq. (11.12) gives

$$p = \frac{E_m I_m}{2} \sin 2\omega t \tag{11.14}$$

This is the equation of the instantaneous-power curve shown graphically in Fig. 11.3*c*. It will be noted that the axis of the power wave coincides with the x-axis. The shaded area above the x-axis represents the power flowing into the inductive reactance and exactly equals the shaded area below the x-axis, which represents power flowing away from the inductive reactance back to the generator. This means that the average dissipated power is zero, because the distance from the x-axis of the instantaneous-power wave is zero and the positive power pulses above the axis exactly cancel the negative power pulses below the axis Again it will be noted that the instantaneous-power wave is double the frequency of the voltage or current wave.

In a pure capacity the current leads the voltage by 90°, and from

Fig. 10.3 we can write

$$e = -E_m \cos \omega t \tag{11.15}$$

$$i = I_m \sin \omega t \tag{11.3}$$

Using the identity of Eq. (11.13) and substituting these values in Eq. (11.1) yields

$$p = -\frac{E_m I_m}{2} \sin 2\omega t \tag{11.16}$$

Owing to the negative sign, this instantaneous-power curve is 180° out of phase with the one in Eq. (11.14). These equations can be used to explain how the instantaneous power can oscillate from the electrostatic field of a capacitor to the electromagnetic field of an inductance when

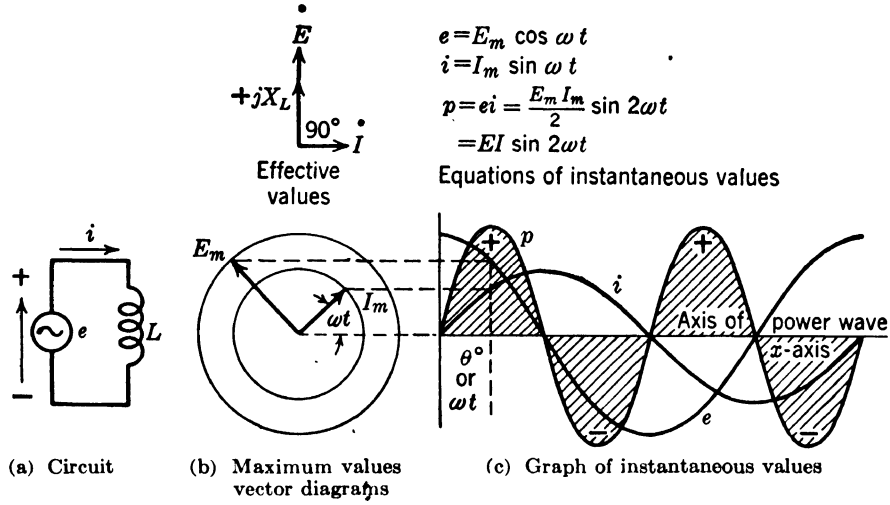


FIG. 11.3. Alternating-current power in a pure inductive reactance.

they are connected in series to make an oscillator. If either an inductance or capacity is connected to a generator, it will absorb power for a half power cycle and feed it back to the generator during the next half cycle.

11.3. Alternating-current Power in a Circuit Containing a Pure Resistance in Series with a Pure Inductance. We will first consider the case of an inductance in series with a resistance. Such a circuit is shown in Fig. 11.4. It will be recalled that the current lags the voltage by 90° in a pure inductance and is in phase with the voltage in a pure resistance. Then in any circuit containing both inductance and resistance, the current will lag the voltage at some angle between 0 and 90°. The angle by which the current lags the voltage is known as the *phase angle* ϕ. If the inductive reactance ($X_L = 2\pi fL$ by Eq. (10.13)) is large, owing to either a high frequency or a large inductance and if the effective resistance R

of the circuit is small, the phase angle ϕ will be large and approach 90° as this ratio $X_L/R \rightarrow \infty$ (approaches infinity). However, if X_L is small and R is large, the phase angle ϕ will be small, and as the ratio $X_L/R \rightarrow 0$, the phase angle ϕ will approach 0°. In order to determine the phase angle ϕ, it is very convenient to use vector algebra and to draw an *impedance triangle* or vector diagram as shown in Fig. 11.4*a*. It has already been shown that the voltage across a pure inductance leads the current through the inductance by 90°. In the case of a current through

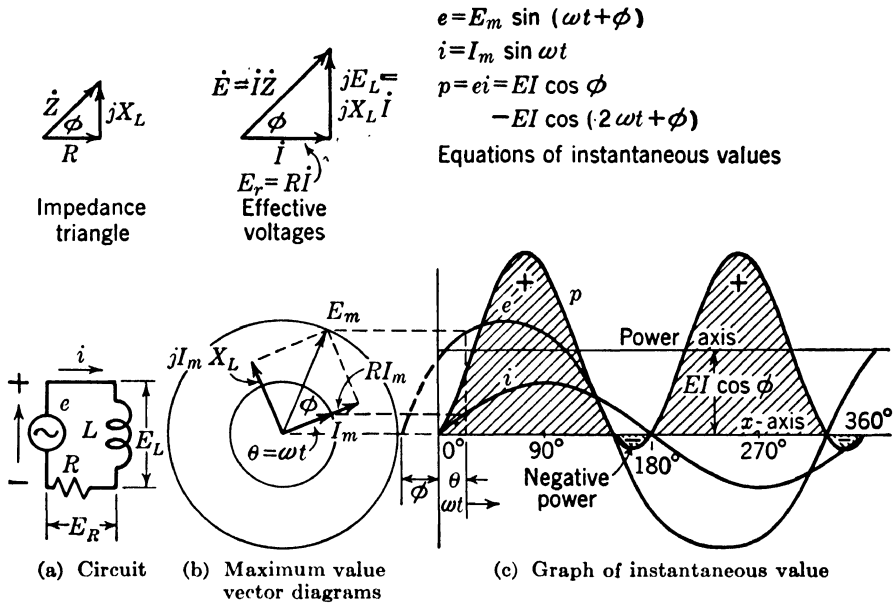


FIG. 11.4. Alternating-current circuit containing resistance and inductance.

an inductance and resistance in series, this will still be true. By Kirchhoff's law, the current around the circuit must be the same at all points at any instant, and the sum of the voltage drops across the elements of the circuit must at any instant equal the supply voltage. By the use of vector algebra, this situation can be visualized graphically in a very simple manner. The impedance of the resistance and inductance in series may be represented as shown in the small triangle at the left in Fig. 11.4, and the voltage relationships may be represented as shown in the larger triangle. If X_L and R are known, the phase angle ϕ can be determined by the following equation:

$$\phi = \tan^{-1} \frac{X_L}{R} \tag{11.17}$$

where ϕ = phase angle
$\tan^{-1}$ = arc tangent (the angle whose tangent is)
X_L = inductive reactance, ohms
R = effective resistance, ohms

Equation (11.17) is used to find the phase angle ϕ when X_L and R are known. It can be written in other forms when other quantities are given; thus

$$\tan \phi = \frac{X_L}{R} \tag{11.18}$$

$$X_L = R \tan \phi \tag{11.19}$$

$$R = \frac{X_L}{\tan \phi} \tag{11.20}$$

Example 11.5. Find the inductive reactance of a load circuit containing 25 ohms effective resistance and having a phase angle of 60°.

Solution. Using Eq. (11.19)

$$X_L = (25)(\tan 60°)$$
$$= (25)(1.732) = 43.3 \text{ ohms} = \text{inductive reactance} \qquad \textit{Ans.}$$

The voltage drop across the resistance R is

$$E_R = IR \tag{11.21}$$

and the voltage drop across the inductive reactance X_L is

$$E_L = IX_L \tag{11.22}$$

Both Eqs. (11.21) and (11.22) are in the form of Ohm's law. That is, a voltage drop is given as the product of the current by a resistance or reactance element. One form of Ohm's law is

$$R = \frac{E}{I}$$

In one sense, this may be taken as *defining* resistance as the quotient of voltage by current. In circuits where the voltage and current are not in phase, it is natural to define some quantity that will be like resistance in that it will be defined by the quotient of voltage by current. In a-c circuits, the quantity thus defined is known as *impedance*. In terms of vector voltages, currents, and impedances, we may write at once

$$\dot{E} = \dot{I}\dot{Z} \tag{11.23}$$

where $\dot{E}$ = effective vector voltage, volts
$\dot{I}$ = effective vector current, amp
$\dot{Z}$ = impedance vector, ohms

This important vector equation is the *generalized form of Ohm's law.* It

is perfectly general in that the impedance can be resistive or can have either an inductive- or capacitive-reactance component. The effective-voltage vector diagram for the above inductive circuit is shown in Fig. 11.4*b*.

When dealing with vector quantities exclusively, the dot above the letter is usually omitted. When most of the quantities in the equation have magnitude only, a dot may be placed over the vector quantities to distinguish them from quantities having magnitude only. When most of the quantities are vector quantities, it is less work to omit the dot above vector quantities and place bars on both sides of the quantities having magnitudes only. With this latter scheme, the magnitude of Z is written $|Z|$.

Example 11.6. If 2 amp of alternating current flows in the load impedance of Example 11.5, determine the voltage by the application of Kirchhoff's law.

Solution. Substituting in Eq. (11.21),

$$E_R = IR = (2)(25) = 50 \text{ volts across resistance}$$

Substituting in Eq. (11.22),

$$E_L = IX_L = (2)(43.3) = 86.6 \text{ volts across the inductance}$$

But since X_L is a pure inductance, the voltage E_L across it leads the voltage across E_R by 90°; hence when we write the vector equation of the voltage drops around the circuit, a j operator must be placed in front of E_L; thus with I as a reference

$$\underline{/I} \qquad \dot{E} = E_R + jE_L = IR + jIX_L$$
$$= 50 + j86.6 = 100\underline{/60°} \qquad \textit{Ans.}$$

The *magnitude* of the impedance vector Z is the length of the hypotenuse of the impedance triangle; hence for an inductive load

$$Z = \sqrt{R^2 + X_L{}^2} \tag{11.24}$$

Or the *impedance vector* in *rectangular form* may be written

$$\dot{Z} = R + jX_L \tag{11.25}$$

and in *polar form* this equation becomes

$$\dot{Z} = \sqrt{R^2 + X_L{}^2}\,\underline{/\tan^{-1}\frac{X_L}{R}} \tag{11.26}$$

This is in the notation developed in Chap. 7 of "Applied Mathematics."

Example 11.7. Determine the load impedance vector of Example 11.5, and with 2 amp flowing, determine the voltage by the application of Ohm's law.

Solution. Substituting in Eq. (11.25), the load impedance vector in rectangular form is

$$\dot{Z} = R + jX_L = 25 + j43.3 \qquad \textit{Ans.}$$

In polar form, by Eq. (11.26), the load impedance vector is

$$\dot{Z} = \sqrt{R^2 + X_L^2} \angle \tan^{-1} \frac{X_L}{R} = \sqrt{(25)^2 + (43.3)^2} \angle \tan^{-1} \frac{43.3}{25}$$

$$= 50/\underline{60°} \qquad \textit{Ans.}$$

By Ohm's law as given in Eq. (11.23)

$$\dot{E} = \dot{I}\dot{Z} = (2/\underline{0°})(50/\underline{60°}) = 100/\underline{60°} \qquad \textit{Ans.}$$

This checks the solution of Example 11.6.

If the maximum voltage and maximum current are used, the revolution of the triangle will generate the instantaneous values as illustrated in Figs. 11.4*b* and *c*. It will be noted that the current lags the voltage by some phase angle ϕ, which is less than 90°. This phase angle ϕ is represented in the graph of instantaneous values by the distance between the time the voltage crosses the x-axis (voltage curve dotted) and the time the current crosses the x-axis. The student should make a careful study of these curves and obtain a clear picture of what is happening. The equation of the instantaneous voltage is

$$e = E_m \sin (\omega t + \phi) \tag{11.27}$$

and the equation of the instantaneous current is

$$i = I_m \sin \omega t \tag{11.3}$$

Substituting for e and i from Eqs. (11.27) and (11.3) in Eq. (11.1) gives

$$p = E_m I_m \sin \omega t \sin (\omega t + \phi) \tag{11.28}$$

Substituting $a = \omega t + \phi$ and $b = \omega t$, the above equation becomes

$$p = E_m I_m \sin a \sin b \tag{11.29}$$

From a trigonometric identity:

$$\sin a \sin b = \frac{\cos (a - b) - \cos (a + b)}{2} \tag{11.30}$$

which substituted in the above equation gives

$$p = \frac{E_m I_m}{2} \cos (a - b) - \frac{E_m I_m}{2} \cos (a + b) \tag{11.31}$$

But $a - b = \phi$ and $a + b = 2\omega t + \phi$, which, if substituted in the above equation, gives

$$p = \frac{E_m I_m}{2} \cos \phi - \frac{E_m I_m}{2} \cos (2\omega t + \phi) \tag{11.32}$$

where p = instantaneous power, watts
E_m = maximum instantaneous voltage, volts
I_m = maximum instantaneous current, amp
ϕ = phase angle between rotating vectors E_m and I_m
$2\omega t$ = angular velocity, radians/sec

Just as in Eq. (11.6), the first term of Eq. (11.32) gives the distance from the x-axis up to the axis of the instantaneous-power wave, as shown graphically for an inductive circuit in Fig. 11.4c. The second term of Eq. (11.32) is the equation of the power wave about its own axis. Again it will be noted that the power wave completes two cycles while the voltage and current complete only one. That is the reason that the number 2 appears in cos $(2\omega t + \phi)$ of Eq. (11.32).

For a pure resistance load, the phase angle between E_m and I_m is zero; hence, substituting this value in Eq. (11.32) makes cos $\phi = 1$ and reduces cos $(2\omega t + \phi)$ to cos $2\omega t$. If these values are substituted in Eq. (11.32), the result is Eq. (11.6).

For a pure inductance, $\phi = 90°$. Substituting this in Eq. (11.32) makes cos $90° = 0$, and the first term goes to zero.

$$\cos (2\omega t + 90°) = -\sin 2\omega t,$$

which reduces Eq. (11.32) to Eq. (11.14). Now, for a pure capacity, $\phi = 90°$. Then cos $(-90°) = 0$, and again the first term of Eq. (11.32) vanishes. Cos $(2\omega t - 90°) = \sin 2\omega t$, which substituted in Eq. (11.32) yields Eq. (11.16).

So far, phase angles of 0°, +90°, and −90° have been tried in Eq. (11.32) with results that check with other equations. All other loads must have phase angles between +90° for a pure inductance and −90° for a pure capacity.

Example 11.8. An inductive load of 86.6 ohms resistance and 50 ohms reactance is connected to a 1,000-cycle generator delivering an effective voltage of 10 volts. Determine the maximum and minimum instantaneous power absorbed by the load. What are corresponding values of ωt?

Solution. By Eq. (11.26) the impedance is

$$\dot{Z} = \sqrt{(86.6)^2 + (50)^2} \angle \tan^{-1} \frac{50}{86.6} = 100\angle 30°$$

The effective current by Ohm's law given in Eq. (11.23) is

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \frac{10\angle 0°}{100\angle 30°} = 0.1\angle -30° \text{ amp}$$

Then

$$\frac{E_m I_m}{2} = \frac{\sqrt{2}\,(10)\,\sqrt{2}\,(0.1)}{2} = 1$$

Eq. (11.32) becomes a maximum when cos $(2\omega t + \phi) = -1$. Substituting these values in Eq. (11.32) gives

$$p_{max} = (1)(\cos 30°) - (1)(-1) = 0.866 + 1 = 1.866 \text{ watts} \qquad \textit{Ans.}$$

Eq. (11.32) becomes a minimum when cos $(2\omega t + \phi) = 1$, giving

$$p_{min} = 0.866 - (1)(1) = -0.134 \text{ watts} \qquad \textit{Ans.}$$

which is the power that is fed back to the generator.

Note: The negative sign means the power is flowing in the opposite direction to that assumed.

For maximum instantaneous power

$$\cos (2\omega t + \phi) = -1$$
$$\cos (2\omega t + 30°) = -1$$

This means that

$$2\omega t + 30° = 180°$$

or $$2\omega t = 180° - 30° = 150°$$

and $$\omega t = 75°$$ *Ans.*

For minimum instantaneous power

$$\cos (2\omega t + \phi) = +1$$
$$\cos (2\omega t + 30°) = +1$$

To meet this condition,

$$2\omega t + 30° = 0°$$

or $$2\omega t = -30°$$

and $$\omega t = -15°$$ *Ans.*

Since a sine wave averages zero over a complete cycle, the average of the instantaneous-power wave about its axis will be zero, and the *average dissipated power* is simply the first term of Eq. (11.32); thus

$$P = \frac{E_m I_m}{2} \cos \phi \tag{11.33}$$

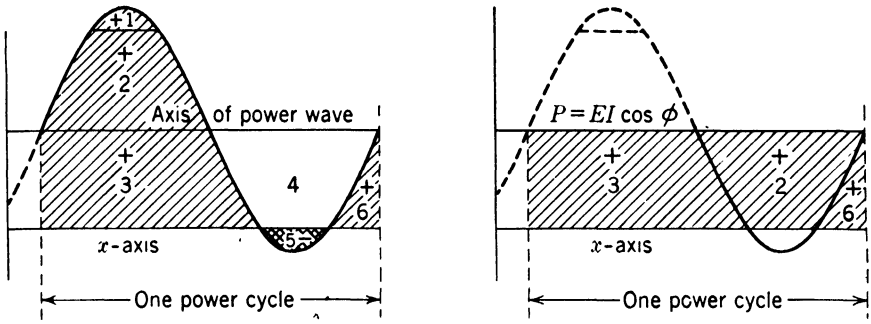
By substituting values from Eqs. (11.8) and (11.9), a more useful form is obtained; thus

$$P = EI \cos \phi \tag{11.34}$$

where P = average dissipated power, watts
E = effective voltage, volts
I = effective current, amp
ϕ = phase angle between E and I, degrees

It will be noted that the curves of Fig. 11.4*c* are different from those of Fig. 11.1*c*, and it is not quite so easy to see from the figure that the average power is represented by the distance from the x-axis to the axis of the power wave. However, this can be illustrated by considering one complete power cycle (it does not matter which one). For convenience, take one complete cycle from Fig. 11.4*c*, starting at the point where the instantaneous-power curve crosses its own axis. Such a portion of the instantaneous-power curve is redrawn in Fig. 11.5*a* with the area between the instantaneous-power curve and the x-axis shaded. The shaded area above the x-axis represents positive power and the shaded area below the x-axis is negative power. During the negative-power portion of the cycle, the load is actually delivering energy rather than absorbing it. If the positive half-cycle area marked 2 and 1 is rotated 180°, it will exactly fill the area marked 4 and 5. The positive area marked 2 will fill the empty area marked 4, and the positive area marked 1 will cancel

the negative area marked 5. The result will be the shaded positive area of Fig. 11.5*b*, which clearly shows that the average ordinate representing the average dissipated power is the distance from the x-axis up to the axis of the power wave.



(a) Instantaneous power wave over a power cycle

(b) Average of instantaneous power wave over a cycle

FIG. 11.5. Averaging the instantaneous power in an inductive circuit.

The average power may also be written

$$P = I^2R \tag{11.35}$$

where P = average dissipated power, watts
I = effective current, amp
R = effective resistance, ohms

Referring to the effective-voltage vector diagrams of Fig. 11.4*b*, it is seen that

$$E_R = IR \tag{11.36}$$

which substituted in Eq. (11.35) gives

$$P = IE_R \tag{11.37}$$

But E_R is the adjacent side of a right triangle. In terms of the hypotenuse of this triangle,

$$E_R = E \cos \phi \tag{11.38}$$

which substituted in Eq. (11.37) gives

$$P = EI \cos \phi \tag{11.39}$$

which is Eq. (11.34) developed by an entirely different method.

Example 11.9. Determine the average power dissipated in the inductive load of Example 11.8 and check the results by a different method.

Solution. Substituting in Eq. (11.34),

$$P = EI \cos \phi = (10)(0.1) \cos 30°$$
$$= (1)(0.866) = 0.866 \text{ watt} \qquad \textit{Ans.}$$

Substituting in Eq. (11.35)

$$P = I^2R = (0.1)^2(86.6) = 0.866 \text{ watt} \qquad \textit{Check}$$

The term cos ϕ is usually spoken of as the *power factor* of the circuit or load. The angle ϕ is in this case sometimes spoken of as the *power-factor angle.*

In the case of a pure resistance, the average power is simply the product of EI, since the voltage and current are in phase. This means that cos $\phi = 1$, or the power factor is unity for a pure resistance. Sometimes the product EI is spoken of as the *apparent power* and is used in rating electrical machinery for two reasons:

1. Eddy currents and hysteresis losses are a function of the magnetic flux and hence the voltage generated.
2. Resistance losses in the winding are a function of the current that flows.

Since both factors cause a *temperature rise,* which is the primary basis of rating a machine, apparent power must be used rather than real power. Ratings of large machinery are usually given in kva, which means kilo-volt-amperes apparent power, instead of kw, which means kilowatts real power.

The real power P divided by the apparent power EI gives the *power factor,* which is merely another form of Eq. (11.34).

$$\frac{P}{EI} = \cos \phi \tag{11.40}$$

This is another way of defining the power factor.

The in-phase component of the voltage E_R, multiplied by the current, gives the real power. Sometimes the out-of-phase voltage E_L, multiplied by the current, is desired. It is called the *reactive power.* In terms of E, the reactive voltage E_L can be written (see the effective-voltage triangle in Fig. 11.4*b*)

$$E_L = E \sin \phi \tag{11.41}$$

Then the reactive power can be written

$$\text{Reactive power} = IE_L = EI \sin \phi \tag{11.42}$$

Sin ϕ is sometimes called the *reactive factor.* By using the following trigonometric identity:

$$\sin^2 \phi + \cos^2 \phi = 1 \tag{11.43}$$

the reactive factor can be written in terms of the power factor; thus, transposing,

$$\sin^2 \phi = 1 - \cos^2 \phi$$

and extracting the square root of both sides,

$$\text{Reactive factor} = \sin \phi = \sqrt{1 - \cos^2 \phi} \tag{11.44}$$

or

$$\text{Reactive factor} = \sqrt{1 - (\text{power factor})^2} \tag{11.45}$$

Example 11.10. Determine the reactive power and reactive factor for the inductive load of Example 11.8.

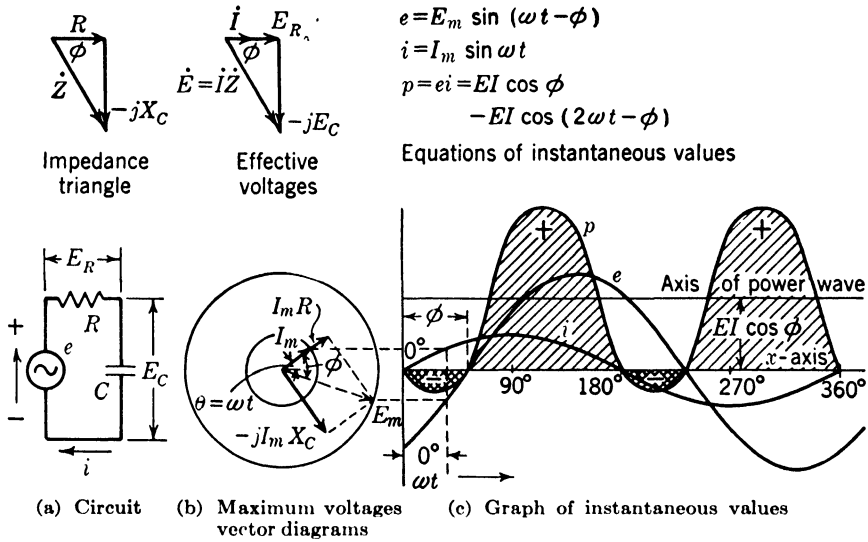
Solution. By Eq. (11.42) the

$$\text{Reactive power} = EI \sin \phi = (10)(0.1)(\sin 30°)$$
$$= (1)(0.5) = 0.5 \text{ wat} \qquad \textit{Ans.}$$

By Eq. (11.45) the

$$\text{Reactive factor} = \sqrt{1 - (\text{power factor})^2}$$
$$= \sqrt{1 - (0.866)^2} = 0.5 \qquad \textit{Ans.}$$
$$\text{Reactive factor} = \sin \phi = \sin 30° = 0.5 \qquad \textit{Check}$$

11.4. Alternating-current Power in a Circuit Containing a Pure Resistance in Series with a Pure Capacity. Such a circuit can be diagramed as in Fig. 11.6*a*. The next case to be considered is a circuit consisting of a capacitor in series with a resistor.



(a) Circuit (b) Maximum voltages vector diagrams (c) Graph of instantaneous values

FIG. 11.6. Alternating-current circuit containing resistance and capacity.

In the case of a circuit containing a pure capacity, the current will *lead* the voltage by 90°, or the voltage is said to *lag* the current by 90°. If resistance is in series with the capacity, the current will lead the voltage at some phase angle between 0 and 90° depending upon the values of resistance and reactance. If the resistance is small and the capacitive reactance large, the phase angle will approach 90°, but if the resistance is large and the capacitive reactance small, the phase angle will be nearly 0°.

To obtain a quantitative idea about the phase angle and impedance of a capacitive circuit, it is convenient to use the impedance triangle or vector diagram as shown in Fig. 11.6*a*. It will be recalled [see Eq.

(10.23)] that the capacitive reactance $X_C = 1/2\pi fC$ is proportional to the reciprocal of the frequency f and capacity C. With X_C and R known, the phase angle ϕ can be determined by Eq. (11.17) if X_C is used instead of X_L. Thus

$$\phi = \tan^{-1} \frac{X_C}{R} \tag{11.46}$$

The *magnitude* of the *impedance vector* $\dot{Z}$ of this circuit is the hypotenuse of the right triangle; hence the magnitude is

$$Z = \sqrt{R^2 + X_C^2} \tag{11.47}$$

which is similar to Eq. (11.24). One difference in the impedance triangle must be noted: the vector X_C is down instead of up; hence in rectangular form, the reactance vector is written with a negative sign

$$\dot{Z} = R - jX_C \tag{11.48}$$

and in polar form, this equation is written with a negative phase angle

$$\dot{Z} = \sqrt{R^2 + X_C^2} \angle - \tan^{-1} \frac{X_C}{R} \tag{11.49}$$

It will be noted that the negative sign in these equations is due to the negative angle the voltage makes with the current.

Referring now to the voltage triangles of Fig. 11.6*b*, it will be noted that the voltage vector is in the same direction as the impedance vector, and the current vector is in the same direction as the resistance vector. The voltage E_C across the reactance X_C is in the same direction as the reactance vector $-jX_C$ of the impedance triangle.

If the maximum-voltage triangle starts to rotate counterclockwise when the current vector lies along the positive x-axis, it will generate the instantaneous values, as shown in Fig. 11.6*c*. The equations of these curves are

$$e = E_m \sin (\omega t - \phi) \tag{11.50}$$

$$i = I_m \sin \omega t \tag{11.3}$$

It will be noted that ϕ is negative, owing to the fact that voltage lags, while in Eq. (11.27) it was positive because the voltage led the current by this phase angle. The current was started from the same position along the positive x-axis and may be considered as the *reference vector*, meaning the vector from which the phase angle is measured.

The equation for the instantaneous power is Eq. (11.32), if negative values of ϕ are used, or if the equation is written like Eq. (11.50) with

$-\phi$, we have

$$p = \frac{E_m I_m}{2} \cos \phi - \frac{E_m I_m}{2} \cos (2\omega t - \phi) \tag{11.51}$$

Since $\cos (-\phi) = \cos \phi$, the first term is unchanged.

The average dissipated power is the first term of this equation, since the second term averages zero over a complete cycle. Hence, as was developed for an inductive load,

$$P = EI \cos \phi \tag{11.34}$$

Referring to the effective-voltage vector diagram of Fig. 11.6*b*, the effective voltage across the capacitive reactance is

$$E_C = IX_C = E \sin \phi \tag{11.52}$$

The reactive power can now be written

$$\text{Reactive power} = IE_C = EI \sin \phi \tag{11.53}$$

and the reactive factor is $\sin \phi$, as it was in the case of an inductive load [see Eqs. (11.44) and (11.45)].

Example 11.11. A 2 μf capacitor is connected in series with a 1,000-ohm resistance and connected to a 25-volt 60-cycle generator. Determine the capacitive reactance of the capacitor, the impedance of the capacitive load, the power factor, and the reactive power. Write the equation of the instantaneous power.

Solution. The capacitive reactance is

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(6.2832)(60)(2 \times 10^{-6})} = 1{,}326 \text{ ohms} \qquad \textit{Ans.}$$

By Eqs. (11.48) and (11.49)

$$Z = R - jX_C = 1{,}000 - j1{,}326 = 1{,}662\underline{/-52°59'} \qquad \textit{Ans.}$$

$$\text{Power factor} = \cos 52°59' = 0.602 \qquad \textit{Ans.}$$

By Ohm's law, as given in Eq. (11.23),

$$\underline{/\dot{E}} \qquad \dot{I} = \frac{\dot{E}}{\dot{Z}} = \frac{25\underline{/0°}}{1{,}662\underline{/-52°59'}} = 0.01504\underline{/52°59'}$$

Then the reactive power by Eq. (11.53) is

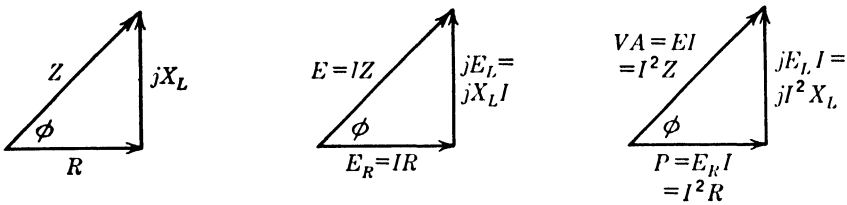
$$\text{Reactive power} = EI \sin \phi = (25)(0.01504)(\sin 52°59') = (0.376)(0.7985) = 0.3002 \text{ watt} \qquad \textit{Ans.}$$

The instantaneous-power equation by Eq. (11.51) is

$$p = \frac{E_m I_m}{2} \cos \phi - \frac{E_m I_m}{2} \cos (2\omega t - \phi) = 0.2262 - 0.376 \cos (754t - 52°59') \qquad \textit{Ans.}$$

11.5. Power Diagram. It is possible to draw a power diagram by referring to the impedance and voltage triangles as shown in Fig. 11.7.

The impedance triangle of Fig. 11.7*a* is for an inductive series circuit. As shown in Fig. 11.7*b* the voltage across the impedance is $E = IZ$, the voltage across the inductive reactance is $jE_L = jX_LI$, and the voltage across the resistance in $E_r = IR$. The power diagram is a similar triangle as shown in Fig. 11.7*c*. In this diagram the apparent power = $EI = I^2Z$; the reactive power often referred to as volt-amperes-reactive, from which the first letters form the word VAR, is



(a) Impedance triangle (b) Voltage triangle (c) Power triangle

FIG. 11.7. Impedance, voltage and power diagrams.

$$\text{Reactive power} = VAR = E_LI = I^2X_L$$

and the real power is expressed

$$P = E_rI = I^2R$$

Summary

The *instantaneous power* in a circuit is always equal to the product of the instantaneous voltage and instantaneous current, that is, $p = ei$. This law is true for any circuit containing reactance as well as resistance, and is true regardless of wave form.

In a circuit consisting of resistance only, the current is always in phase with the voltage, and the power factor is always unity. In such a circuit the average power is equal to one-half the peak power and is equal to the product of the effective voltage and effective current.

In a circuit consisting of pure reactance, the current is 90° out of phase with the applied voltage. The average power is always zero for a pure reactance, and the power factor is zero.

For a circuit consisting of inductance and resistance, the phase angle between the current and applied voltage will be between 0 and 90°, the current lagging the voltage. The phase angle can be determined from the relation

$$\phi = \tan^{-1}\frac{X_L}{R}$$

The total impedance of the circuit is given by the relationship

$$Z = \sqrt{R^2 + X_L^2}$$

and the current is given by

$$I = \frac{E}{Z}$$

The average power absorbed can be determined from several equivalent formulas:

$$P = I^2R = EI \cos \phi = \frac{E_R^2}{R}$$

The apparent power is obtained as the product of voltage and current, but is *not* the actual power in the circuit. The product $EI \sin \phi$ is called the *reactive power* and is not useful power.

When a pure capacitance is connected in series with a resistance, the same relationships hold as with a resistance in series with an inductance. The only difference in this case is that the reactance is negative, and the current leads the voltage. The mathematical relationships are identical except for the minus sign in front of the reactance term.

Exercises

11.1. Define average power, phase angle, power factor, real power, apparent power, and reactive factor.

11.2. How is a-c machinery rated? Explain why.

11.3. Given a resistance load where $\omega = 5{,}000$ radians/sec, $E = 250$ μv, $R = 500$ ohms, (*a*) write the equations of the instantaneous voltage, current, and power. (*b*) What is the frequency of the power wave?

11.4. In Exercise 11.3, (*a*) determine the impedance, phase angle, and reactive factor. (*b*) Determine the effective current and draw a voltage vector diagram to scale. (*c*) Determine the real power in watts.

11.5. Does the load of Exercise 11.3 ever feed energy back into the line? Explain why?

11.6. If a quarter-wave antenna has an effective radiation resistance of 36.7 ohms, what will be the effective and maximum voltage and current when it is radiating 50 kw?

11.7. A load draws 0.6 amp when 100-volt, 60-cycle alternating current is applied. If the load has an inductance of 100 mh, (*a*) what is its resistance? (*b*) What is the power factor of the load?

11.8. In Exercise 11.7 draw the impedance triangle of the load to scale. Superimpose upon it the voltage vectors of the circuit. What is the voltage across the inductance and across the resistance?

11.9. The effective voltage across a 25-ohm resistor is 10 volts. If this resistor is in series with an 8-μf capacitor in a 60-cycle a-c circuit, what is the voltage across the capacitor and across the two elements in series?

11.10. (*a*) What is the impedance of the load of Exercise 11.9? (*b*) What is the apparent power, power factor, and reactive factor of this circuit?

CHAPTER 12

ALTERNATING-CURRENT SERIES CIRCUITS

When two parts of a circuit are so connected that the same current flows through each part they are said to be *in series.* In the last chapter, series circuits containing resistance and inductance in series and resistance and capacity in series were treated. In general, the voltage drops across the various elements are not equal nor will they be in phase. In this chapter, the phenomena of series resonance in a-c series circuits containing more than two elements will be treated.

12.1. Series Resonance Phenomena. Consider the series circuit illustrated in Fig. 12.1. The impedance vector of the series circuit can now be written

$$Z = R + j(X_L - X_C) \tag{12.1}$$

where Z = impedance vector, ohms
R = resistance, ohms
$X_L = 2\pi fL$ = inductive reactance, ohms
$X_C = \dfrac{1}{2\pi fC}$ = capacitive reactance, ohms

This is the equation of the vector impedance in rectangular form. In this type of equation, it is convenient to think of all magnitude quantities (quantities without the dot) as lying along the positive x-axis. When such a quantity is multiplied by the j operator, it is rotated counterclockwise 90° and lies along the positive y-axis.

If Eq. (11.1) is changed to polar form, it can be written

$$\dot{Z} = Z\underline{/\phi} \tag{12.2}$$

where Z = impedance vector, ohms
$Z = \sqrt{R^2 + (X_L - X_C)^2}$ = the magnitude of the impedance vector, ohms
$\phi = \underline{/\tan^{-1}\dfrac{(X_L - X_C)}{R}}$ = the phase angle of the impedance vector

Example 12.1. Given a series circuit, as shown in Fig. 12.1, with

$L = 2$ mh
$R = 40$ ohms
$C = 2$ μf
$E = 10$ volts

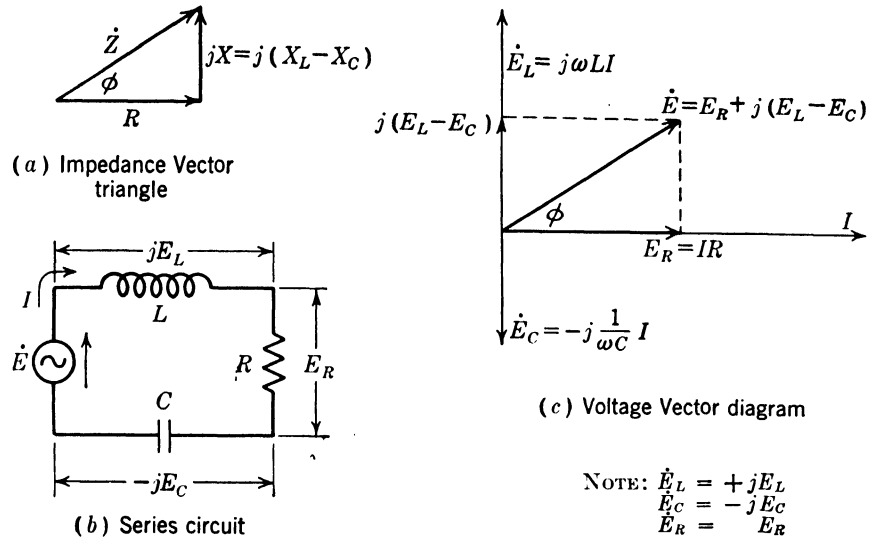


FIG. 12.1. Sine-wave voltage imposed on a simple series circuit containing inductance, resistance, and capacity.

Determine $\dot{Z}$ in rectangular and polar form when the frequency $f = 1{,}592$ cycles/sec What are the voltages across the various elements? Draw a vector diagram to scale.

Solution. Solving for the inductive reactance

$$\begin{aligned} X_L &= 2\pi fL = \omega L \\ &= 2\pi(1{,}592)(0.002) = (10{,}000)(0.002) \\ &= 20 \text{ ohms} = \text{inductive reactance} \end{aligned}$$

The capacitive reactance is

$$\begin{aligned} X_C &= \frac{1}{2\pi fC} = \frac{1}{\omega C} \\ &= \frac{1}{10{,}000C} = \frac{10^6}{(10{,}000)(2)} \\ &= 50 \text{ ohms} = \text{capacitive reactance} \end{aligned}$$

From Eq. (12.1) the impedance in *rectangular form* can be written

$$\dot{Z} = 40 + j(20 - 50) = 40 - j30 \qquad \textit{Ans.}$$

Changing this to the *polar form* by Eq. (12.2) gives

$$\begin{aligned} \dot{Z} &= \sqrt{40^2 + 30^2} \,\underline{/\tan^{-1}\frac{20-50}{40}} \\ &= 50\underline{/\tan^{-1}(-0.75)} \\ &= 50\underline{/-36°52'} \end{aligned} \qquad \textit{Ans.}$$

Sometimes it is desirable to change from polar to rectangular form. To do this, use the following equation, which has been given before.[1]

[1] Smith, Carl E., "Applied Mathematics," 1st ed., McGraw-Hill, New York, 1945, p. 94.

$$\dot{Z} = Z(\cos\phi + j\sin\phi)$$

Substituting $\dot{Z} = (50)(0.8 - j0.6)$

$= 40 - j30$ *Check*

Note: The sine of a negative angle up to 180° is negative. Hence

$$\sin(-36°52') = -0.6$$

Since the same current flows through each element, this current can be used to determine the voltages across the various elements. This current can be determined from the most general form of Ohm's law; thus

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \frac{10\angle\phi}{50\angle -36°52'}$$

If the vector current $\dot{I}$ is to lie along the x-axis, the angle that $\dot{E}$ makes with this axis must be the same as the angle that $\dot{Z}$ makes with the axis. This can be visualized from the vector diagram of this problem shown in Fig. 12.2. Then the preceding equation can be written

$\angle I$ $$I = \frac{10\angle -36°52'}{50\angle -36°52'} = 0.2\angle 0° \text{ amp} \qquad \text{(in polar form)}$$

or $$\dot{I} = 0.2 + j0 \text{ amp} \qquad \text{(in rectangular form)}$$

The voltage drop across the inductive reactance is

$\angle I$ $$\dot{E}_L = I(jX_L) = (0.2\angle 0°)(20\angle +90°)$$

$$= 4\angle +90° \qquad \text{(in polar form)}$$

$$= 0 + j4 \qquad \text{(in rectangular form)} \qquad \textit{Ans.}$$

which means that the voltage across the inductance leads the current by 90°.

The voltage drop across the capacitive reactance is

$\angle I$ $$\dot{E}_C = I(-jX_C) = (0.2\angle 0°)(50\angle -90$$

$$= 10\angle -90° \qquad \text{(in polar form)}$$

$$= 0 - j10 \qquad \text{(in rectangular form)} \qquad \textit{Ans.}$$

which means that the voltage across the capacity lags the current by 90°.

The voltage drop across the resistance is

$\angle I$ $$\dot{E}_R = IR = (0.2\angle 0°)(40\angle 0°)$$

$$= 8\angle 0° \qquad \text{(in polar form)}$$

$$= 8 + j0 \qquad \text{(in rectangular form)} \qquad \textit{Ans.}$$

which means that this voltage is in phase with the current.

The vector diagram is given in Fig. 12.2. *Ans.*

If the frequency of the circuit in Example 12.1 were increased, the inductive reactance would increase, and the capacitive reactance would decrease until finally they would be equal, and the resulting reactance of the circuit would be zero. This is known as *series resonance.* Equation (12.1) would then be reduced to

$$Z = R \qquad \text{[Eq. (12.1) for series resonance]} \qquad (12.3)$$

The frequency at series resonance can easily be determined from the fact that

$$X_L = X_C \qquad \text{(for series resonance)} \tag{12.4}$$

Substituting the reactance values,

$$2\pi f_r L = \frac{1}{2\pi f_r C} \tag{12.5}$$

Solving for f_r

$$f_r = \frac{1}{2\pi\sqrt{LC}} \tag{12.6}$$

where f_r = resonant frequency
$2\pi = 6.2832$
L = inductance, henrys
C = capacity, farads

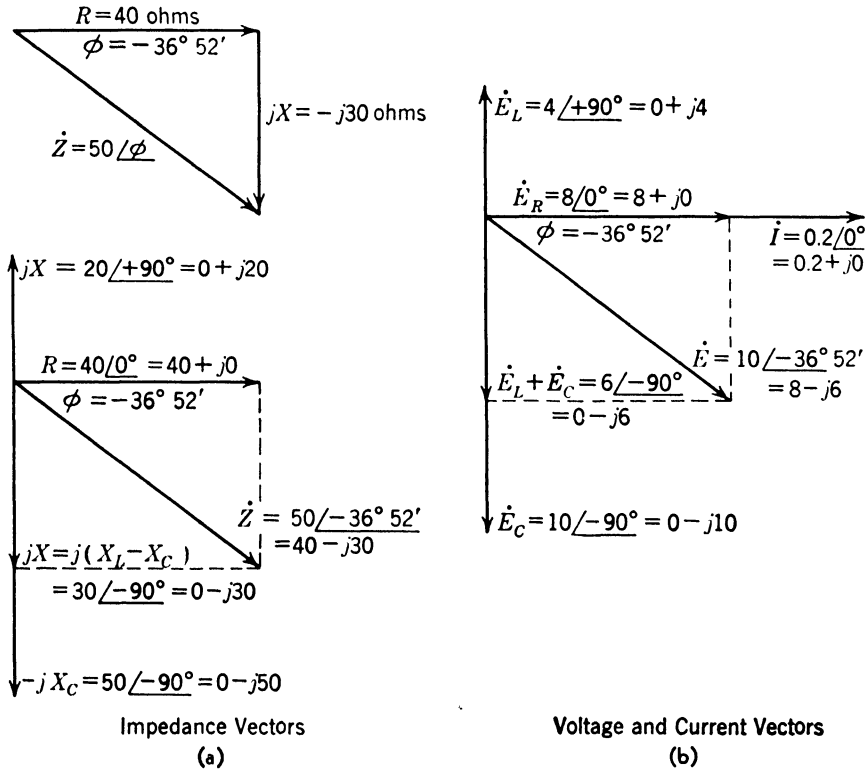


FIG. 12.2. Vector diagrams for solution of Example 12.1.

Example 12.2. Find the frequency at series resonance in Example 12.1. Solve for the series-resonant current and the voltages across the various elements and draw a vector diagram to scale.

Solution. Substituting in Eq. (12.6) gives

$$f_r = \frac{1}{6.2832\sqrt{(0.002)(2 \times 10^{-6})}}$$
$$= 2{,}515 \text{ cycles/sec} = \text{the resonant frequency} \qquad \textit{Ans.}$$

At the resonant frequency, the inductive reactance is

$$X_L = 2\pi f_r L = (6.2832)(2{,}515)(2 \times 10^{-3}) = 316.1 \text{ ohms}$$

And the capacitive reactance is

$$X_C = \frac{1}{2\pi f_r C} = \frac{1}{(6.2832)(2{,}515)(2 \times 10^{-6})} = 316.1 \text{ ohms}$$

It will be noted that these values are equal and will check in Eq. (12.4) or (12.5).

The current flowing is

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \frac{10/0^\circ}{40/0^\circ} = 0.25/0^\circ \text{ amp}$$
$$= 0.25 + j0 \text{ amp} \qquad \textit{Ans.}$$

The voltage drop across the inductive reactance is

$$\dot{E}_L = \dot{I}(jX_L) = (0.25)(j316.1) = j79.05 \text{ ohms}$$
$$= 79.05/90^\circ \text{ ohms} \qquad \textit{Ans.}$$

The voltage drop across the capacitive reactance is

$$\dot{E}_C = \dot{I}(-jX_C) = (0.25)(-j316.1) = -j79.05 \text{ ohms}$$
$$= 79.05/-90^\circ \text{ ohms} \qquad \textit{Ans.}$$

The voltage drop across the resistance is

$$\dot{E}_R = \dot{I}R = (0.25)(40) = 10/0^\circ \text{ volts} \qquad \textit{Ans.}$$

The vector voltage $\dot{E}$ across all the elements in series is simply

$$\dot{E} = \dot{E}_R + \dot{E}_L + \dot{E}_C = 10 + j79.05 - j79.05 = 10 + j0$$
$$\therefore \quad = 10/0^\circ \qquad \textit{Ans.}$$

The circuit and vector diagram drawn to scale is shown in Fig. 12.3. *Ans.*

Example 12.2 illustrates that the voltage across some of the elements in a series circuit can be considerably higher than the total voltage across all the elements in series. In this example, the voltage across the coil is 79.05 volts but is 180° out of phase with the 79.05 volts across the capacitor; hence they completely cancel each other, leaving a 10-volt drop across the resistance, which is also the voltage drop across all three elements in series.

12.2. Loci of Impedance and Admittance Vectors of a Series Circuit. If the frequency is continuously increased in a series circuit from a value below resonance to a value above resonance, the terminal point of the impedance vector

$$\dot{Z} = R + j\left(2\pi fL - \frac{1}{2\pi fC}\right) \qquad \text{[see Eq. (12.1)]} \qquad (12.7)$$

will trace out a straight line perpendicular to the x-axis, as shown in Fig. 12.4*a*. By selecting the correct frequency, any point on this line can be obtained, or by selecting a point on the line, the frequency can

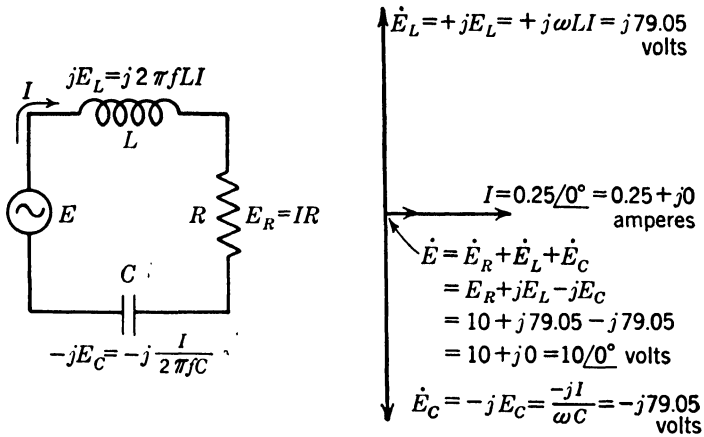
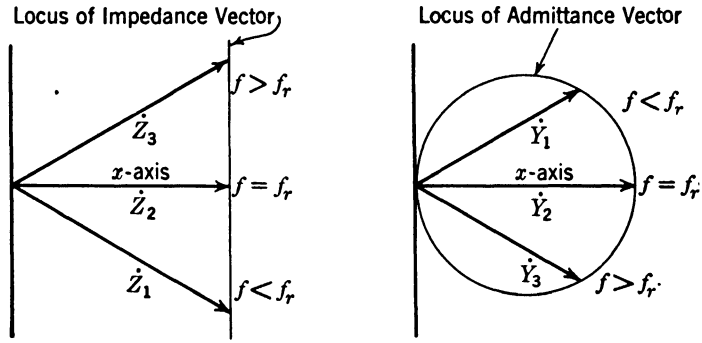


FIG. 12.3. Circuit and voltage vector diagram of the series circuit of Example 12.2 in series resonance.



(a) Impedance vectors (b) Admittance vectors

FIG. 12.4. Loci of impedance and admittance vectors of a series circuit as the frequency is varied above and below the resonant frequency.

be determined. If the reactance X is the ordinate value on the line, the reactance term of Eqs. (12.1) and (12.7) is the same value; hence

$$X = X_L - X_C = 2\pi fL - \frac{1}{2\pi fC} \tag{12.8}$$

Reducing this equation to the general quadratic-equation form by multiplying through by $f/2\pi L$ and rearranging terms gives

$$f^2 - \frac{X}{2\pi L} f - \frac{1}{4\pi^2 LC} = 0 \tag{12.9}$$

Solving this equation for the frequency by the quadratic formula and rearranging the equation results in

$$f = \frac{X}{4\pi L}\left(1 \pm \sqrt{1 + \frac{4L}{X^2C}}\right) \tag{12.10}$$

where f = frequency, cycles/sec
X = total reactance of the series circuit (use + if inductive; use − if capacitive)
L = inductance, henrys
C = capacity, farads

The *reciprocal of the impedance* is called the *admittance;* thus

$$\dot{Y} = \frac{1}{\dot{Z}} \tag{12.11}$$

where $\dot{Y}$ = admittance vector, mhos
$\dot{Z}$ = impedance vector, ohms

The preceding statement and Eq. (12.11) are a definition of admittance. The admittance vector $\dot{Y}$ for corresponding impedance vectors will trace out a circle as shown in Fig. 12.4*b*. It will be noted that Z_1, for a frequency $f < f_r$ (f less than the resonant frequency f_r), is below the x-axis while Y_1 is above the x-axis. The reason for this can easily be shown in polar coordinates as follows:

$$\dot{Z}_1 = Z_1\underline{/-\phi}$$

but

$$\dot{Y}_1 = \frac{1}{\dot{Z}_1} = \frac{1}{Z_1\underline{/-\phi}} = Y_1\frac{1}{\underline{/-\phi}} = Y_1\underline{/\phi}$$

The admittance vector is obtained by taking the reciprocal of the magnitude of the impedance vector and using the phase angle of the impedance vector with the opposite sign.

Example 12.3. An a-c series-resonant circuit has a resistance, inductive reactance, and capacitive reactance of equal magnitude of 22 ohms. If the frequency is reduced to 41.1 per cent of its value at resonance, determine the resultant impedance at the new frequency.

Solution. Multiplying the frequency of Eq. (12.7) by 0.411 to get the new frequency and substituting the values as given above yields

$$\begin{aligned}\dot{Z} &= 22 + j(22)(0.411) - j\frac{22}{0.411}\\ &= 22 + j9.042 - j53.528\\ &= 22 - j44.486 = 49.63\underline{/-65°41'}\end{aligned}$$ *Ans.*

12.3. Reactance Curves of a Series Circuit. The total reactance X of a series circuit can be obtained from Eq. (12.8). To illustrate this

curve graphically, the inductive reactance X_L can be plotted as a function of the frequency f. Since X_L is directly proportional to f, the result will be a straight line sloping up and to the right as shown in Fig. 12.5*a*. Now if the capacitive reactance X_C is plotted as a function of the frequency f, it will be found that X_C is inversely proportional to f, resulting in the curve shown in Fig. 12.5*b*. These curves can be added algebraically as in Eq. (12.8) or graphically, as shown in Fig. 12.5*c*. It will be noted that when $X = 0$, $X_L = X_C$, which gives series resonance and

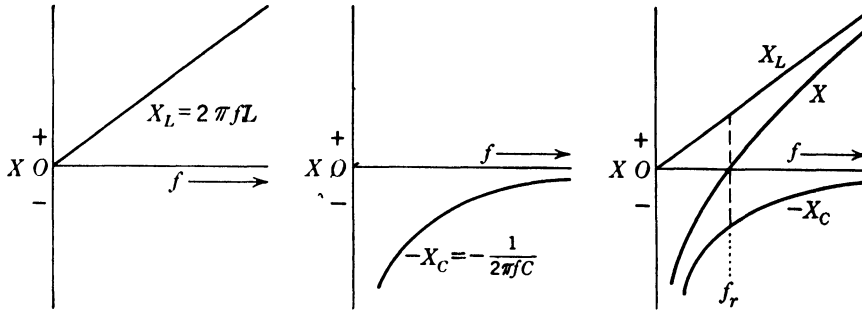


FIG. 12.5. Reactance curves for a series circuit (variable f).

means that the distance to the inductive-reactance curve above the x-axis is equal to the distance from the x-axis to the capacitive-reactance curve below the x-axis.

In a vector diagram, as shown in Fig. 12.4*a*, the distance above and below the x-axis to these curves would be represented by vectors parallel to the positive y-axis and negative y-axis. For instance, frequencies below the resonant frequency result in a small X_L and a large X_C, giving a resulting X that is a capacitive reactance. At resonance, they become equal, and for frequencies above resonance, the resulting X will be an inductive reactance.

Reactance curves are very useful in the study of complex circuits. Much useful information can be obtained from such curves without solving for the exact quantities in the circuit. These curves are especially useful in filter circuit theory to be taken up later.

Example 12.4. Determine the frequency for the series circuit of Example 12.1 when the total reactance is $X = 10$ ohms inductive. What is the value of the impedance, admittance, and current?

Solution. Substituting in Eq. (12.10)

$$f = \frac{10}{(12.566)(2 \times 10^{-3})}\left[1 \pm \sqrt{1 + \frac{(4)(2 \times 10^{-3})}{(10^2)(2 \times 10^{-6})}}\right]$$

$$= 397.9(1 \pm \sqrt{41}) = (397.9)(7.403)$$

$$= 2{,}946 \text{ cycles/sec} \qquad \textit{Ans.}$$

Note: The + sign is used; otherwise the frequency would come out negative, which has no practical meaning.

Checking this answer by substituting in Eq. (12.8) gives

$$X = 2\pi(2{,}946)(2 \times 10^{-3}) - \frac{1}{2\pi(2{,}946)(2 \times 10^{-6})} = 37.01 - 27.01$$
$$= 10 \text{ ohms} = \text{total inductive reactance} \qquad \textit{Check}$$

The impedance by Eq. (12.1) is

$$\dot{Z} = 40 + j(37.01 - 27.01) = 40 + j10 \quad \text{(rectangular form)}$$
$$= 41.23/\underline{14°2'} \quad \text{(polar form)} \qquad \textit{Ans.}$$

The admittance by Eq. (12.11) is

$$\dot{Y} = \frac{1}{41.23/\underline{14°2'}} = 0.02425/\underline{-14°2'} \quad \text{(polar form)}$$
$$= (0.02425)(0.9702 - j0.2425) = 0.02353 - j0.00588 \quad \text{(rectangular form)} \qquad \textit{Ans.}$$

By Ohm's law the current is

$$\dot{I} = \frac{\dot{E}}{\dot{Z}} = \dot{E}\dot{Y} = 10/\underline{0°}(0.02425/\underline{-14°2'})$$
$$= 0.2425/\underline{-14°2'} = 0.2353 - j0.0588 \qquad \textit{Ans.}$$

12.4. Resonance Curves for Series Circuits. If the admittance Y of a series circuit is plotted as a function of frequency f, a resonance curve results. Substituting Eq. (12.1) into Eq. (12.11) gives the vector admittance

$$\dot{Y} = \frac{1}{R + j(X_L - X_C)} \tag{12.12}$$

Since only the absolute magnitude is plotted, we can write

$$Y = \frac{1}{\sqrt{R^2 + X^2}} = \frac{1}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{12.13}$$

or if the values of X_L and X_C are substituted,

$$Y = \frac{1}{\sqrt{R^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}} \tag{12.14}$$

An inspection of Ohm's law shows that the current of a series circuit is proportional to Y and hence gives a resonance curve; thus

$$I = \frac{E}{Z} = EY = \frac{E}{\sqrt{R^2 + \left(2\pi fL - \dfrac{1}{2\pi fC}\right)^2}} \tag{12.15}$$

where I = effective current, amp
E = effective voltage, volts
Z = impedance magnitude, ohms
Y = admittance magnitude, mhos
R = resistance, ohms
$X = X_L - X_C$ = the total reactance, ohms
f = frequency, cycles/sec
L = inductance, henrys
C = capacity, farads

Equations (12.14) and (12.15) are in convenient form for plotting resonance curves. Near resonance, however, the total reactance X has a small value, resulting from taking the difference between two large numbers. To obtain an accuracy of better than 1 per cent in this region, six-place logarithms should ordinarily be used.

In ordinary resonant circuits containing a coil and capacitor, practically all the resistance is in the coil. It is convenient to use the ratio of the inductive reactance to the resistance; this ratio is called the Q of the circuit, defined mathematically as

$$Q = \frac{X_L}{R} = \frac{\omega L}{R} = \frac{2\pi f L}{R} \tag{12.16}$$

The Q of a series circuit can be used to determine most of its characteristics. At *resonance*, the current $I = E/R$, and $X_L = X_C$; hence the voltage across the inductance is

$$E_L = IX_L = \frac{EX_L}{R} = EQ \tag{12.17}$$

and similarly, the voltage across the capacitor is

$$E_C = IX_C = \frac{EX_C}{R} = EQ \tag{12.18}$$

These equations show that there is a resonant rise of Q times the applied voltage across each reactance element at resonance.

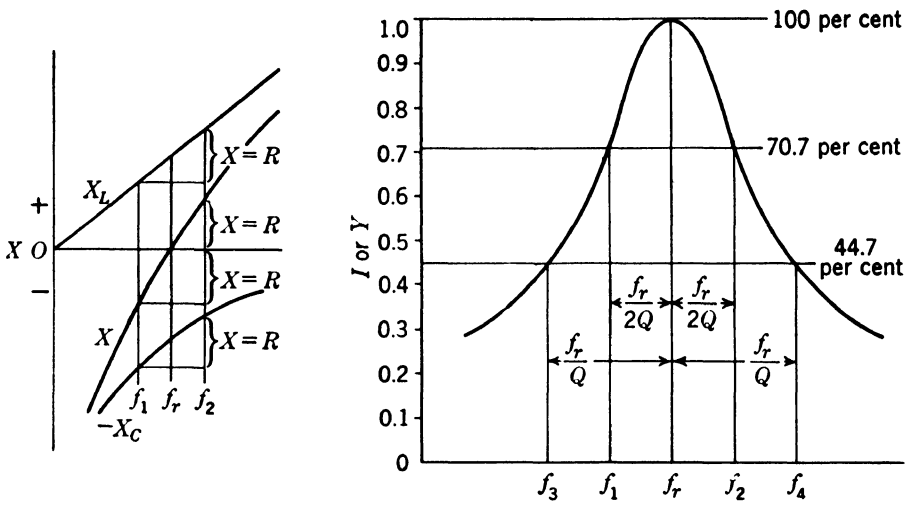
Perhaps the most convenient way to specify the sharpness of resonance is in terms of Q. Another factor, known as the *percentage of frequency discrimination*, can be defined in terms of Q and is the best single factor to indicate this sharpness of resonance. This factor with other points on the resonance curve will now be treated.

The slope of the reactance curve X_L, as shown in Fig. 12.6*a*, is a constant at all frequencies because the curve is a straight line. The slope of X_L can be obtained by forming a triangle as shown in Fig. 12.6*a* and

taking the ratio of the opposite to the adjacent side; thus

$$\text{Slope of } X_L = \frac{2\pi f_2 L - 2\pi f_1 L}{f_2 - f_1} = 2\pi L \tag{12.19}$$

The slope of the reactance curve X_C as shown in Fig. 12.6*a* is not a constant; hence a small triangle must be made at the point of interest to determine the slope at that point. Since we are interested in the slope at f_r, select f_1 below f_r and f_2 above f_r to form a triangle as shown in



(a) Reactance curves (b) Resonance curve

FIG. 12.6. Reactance and resonance curves of a series circuit.

Fig. 12.6*a*. The slope of X_C at the resonant frequency f_r can then be written as the ratio of the opposite to the adjacent side; thus

$$\text{Slope of } X_C = \frac{\left(-\frac{1}{2\pi f_2 C}\right) - \left(-\frac{1}{2\pi f_1 C}\right)}{f_2 - f_1} = -\frac{1}{2\pi f_1 f_2 C} \tag{12.20}$$

Since f_1, f_2, and f_r are approximately equal, f_r can be written as the geometric mean of f_1 and f_2; thus

$$f_r^2 = f_1 f_2 \tag{12.21}$$

Substituting this in Eq. (12.20) gives

$$\text{Slope of } X_C = \frac{1}{2\pi f_r^2 C} \tag{12.22}$$

Since at resonance the reactances are equal,

$$2\pi f_r L = \frac{1}{2\pi f_r C} \tag{12.5}$$

If this equation is divided by f_r, it is seen that the slopes of X_L and X_C, as given in Eqs. (12.19) and (12.22), are equal at this frequency; thus

$$2\pi L = \frac{1}{2\pi f_r^2 C} \tag{12.23}$$

The slope of the total-reactance curve X is then the sum of these two curves of equal slope, giving

$$\text{Slope of } X = 4\pi L \tag{12.24}$$

Now if f_1 is taken as the lower frequency at which the resonant power is reduced to half, and f_2 is the upper frequency at which the resonant power is reduced to half, then

$$R = X \qquad (\text{at } f_1 \text{ and } f_2) \tag{12.25}$$

and the slope of X as shown in Fig. 12.6*a* is

$$\text{Slope of } X = 4\pi L = \frac{2X}{f_2 - f_1} = \frac{2R}{f_2 - f_1} \tag{12.26}$$

Combining this equation with Eq. (12.16), the *frequency discrimination* is

$$\frac{\Delta f}{f_r} = \frac{f_2 - f_1}{f_r} = \frac{R}{2\pi f_r L} = \frac{1}{Q} \tag{12.27}$$

where $\Delta f = f_2 - f_1$ the band width of the circuit
f_2 = upper frequency of half resonant power
f_1 = lower frequency of half resonant power
f_r = resonant frequency
R = resistance of complete series circuit including the generator
L = inductance of the series circuit
Q = Q of coil

Hence $$\frac{100}{Q} = \text{per cent frequency discrimination} \tag{12.28}$$

From this equation two points can easily be obtained on the resonance curve. The resonant frequency f_r is approximately midway between f_1 and f_2; hence with Eq. (12.27)

$$\frac{\Delta f}{2} = \frac{f_2 - f_1}{2} = f_2 - f_r = f_r - f_1 = \frac{f_r}{2Q} \tag{12.29}$$

Then $$f_1 = f_r - \frac{f_r}{2Q} \tag{12.30}$$

and $$f_2 = f_r + \frac{f_r}{2Q} \tag{12.31}$$

These last two equations are very useful in plotting resonance curves, as

shown in Fig. 12.6*b*. They show that when the frequency deviates by $1/2Q$ from the resonant frequency, the power is reduced to 50 per cent of the resonant power, the current to 70.7 per cent of the resonant current, and the current is 45° out of phase with the applied voltage.

Another useful set of points is f_3 and f_4, as shown in Fig. 12.6*b*, where

$$f_3 = f_r - \frac{f_r}{Q} \tag{12.32}$$

$$f_4 = f_r + \frac{f_r}{Q} \tag{12.33}$$

For this deviation of $1/Q$ from the resonant frequency of f_r, the power is reduced to 20 per cent of the resonant power, the current to 44.7 per cent of the resonant current, and the current is 63.5° out of phase with the applied voltage.

The shape of the resonance curve, as shown in Fig. 12.6*b*, can be approximately determined by these five points, which are given as functions of Q. If a more detailed resonance curve is desired, other points

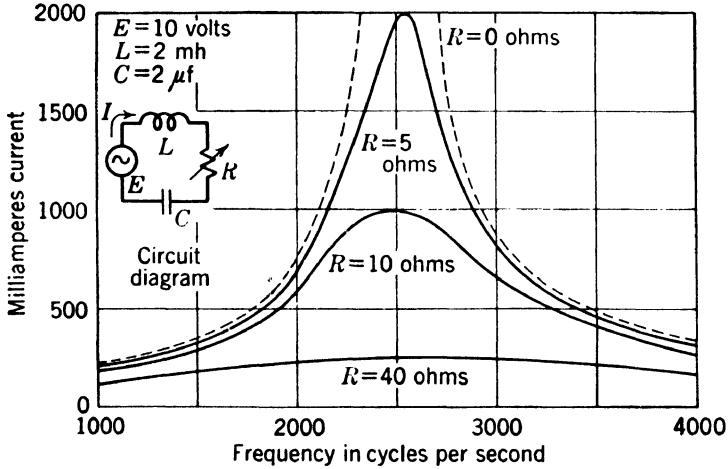


FIG. 12.7. Resonance curves for a series circuit containing different resistances.

can be added by the use of Eq. (12.14) or (12.15). Figure 12.7 shows several resonance curves, indicating how the resistance of a circuit affects the sharpness of the curve. The discrimination of the circuit against frequencies near the resonant frequency is increased very much as the resistance is made small.

The process of varying the frequency to produce resonance is called *tuning* the circuit. Usually it is desired to tune a circuit at a fixed frequency, so it is accomplished by varying either L or C or both. Curves

similar to those in Fig. 12.7 will be obtained with L or C plotted along the x-axis instead of f.

Example 12.5. If the resistance of Example 12.1 is changed to 0.5 ohm and the voltage to 1 volt, determine the Q of the circuit at resonance (generator resistance = zero), the voltage across the capacitor at resonance, the frequency discrimination. Determine five points, with their associated phase angles, to find the approximate resonant curve of the current.

Solution. By Eq. (12.16) and the results from Example 12.2,

$$Q = \frac{2\pi f_r L}{R} = \frac{(6.2832)(2{,}515)(2 \times 10^{-3})}{0.5} = 63.25 \qquad Ans.$$

By Eq. (12.18)

$$E_C = QE = (63.25)(1) = 63.25 \text{ volts} \qquad Ans.$$

By Eq. (12.27), the frequency discrimination is

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q} = \frac{1}{63.25} = 0.01581 \qquad Ans.$$

By Eqs. (12.6), (12.30), (12.31), (12.32), and (12.33), the frequencies for the five points on the resonance curve are found to occur at (with four-place accuracy)

$$f_r = 2{,}515 \qquad Ans.$$

$$f_1 = 2{,}515 - \frac{2{,}515}{(2)(63.25)} = 2{,}495 \qquad Ans.$$

$$f_2 = 2{,}515 + 19.87 = 2{,}535 \qquad Ans.$$

$$f_3 = 2{,}515 - \frac{2{,}515}{63.25} = 2{,}475 \qquad Ans.$$

$$f_4 = 2{,}515 + 39.75 = 2{,}555 \qquad Ans.$$

The currents, taken with respect to the generator voltage, are

$$\dot{I}_r = \frac{E}{R} = \frac{1}{0.5} = 2/0° \text{ amp} \quad (\text{at } f_r) \qquad Ans.$$

$$\dot{I}_1 = (0.707)(2)/+45° = 1.414/+45° \quad (\text{at } f_1) \qquad Ans.$$

$$\dot{I}_2 = (0.707)(2)/-45° = 1.414/-45° \quad (\text{at } f_2) \qquad Ans.$$

$$\dot{I}_3 = (0.447)(2)/+63.5° = 0.894/+63.5° \quad (\text{at } f_3) \qquad Ans.$$

$$\dot{I}_4 = (0.447)(2)/-63.5° = 0.894/-63.5° \quad (\text{at } f_4) \qquad Ans.$$

12.5. Optimum L/C Ratio. The *product* of L and C is used to determine the *resonant frequency*, but it does not fix the individual values of L and C. For example, the inductance can be doubled and the capacity halved without altering the resonant frequency or the product LC, but the ratio L/C will be four times as large. These curves are drawn in Fig. 12.8 for Example 12.1 ($j\omega L$, $-j/\omega C$, X) and also for the case of twice this inductance and half this capacity ($j\omega L'$, $-j/\omega C'$, X').

Selectivity is a measure of the ability of a circuit to tune to a particular wavelength to the exclusion of others. This can be measured by the slope of the reactance curves X and X' in Fig. 12.8. It will be noted that

the curve X' has a larger slope (is steeper) than curve X as they pass through the resonant frequency. This means that the selectivity increases as the Q of the coil increases, and if it were not for the inherent resistance in a series circuit, there would be no limit to the extent to which a circuit's selectivity could be increased. As the L/C ratio is increased, the capacity C becomes smaller, and if the ratio is increased too far, every slight change in capacity will detune the circuit. Usually the mechanical design of variable capacitors determines the capacity range and hence the L/C ratio.

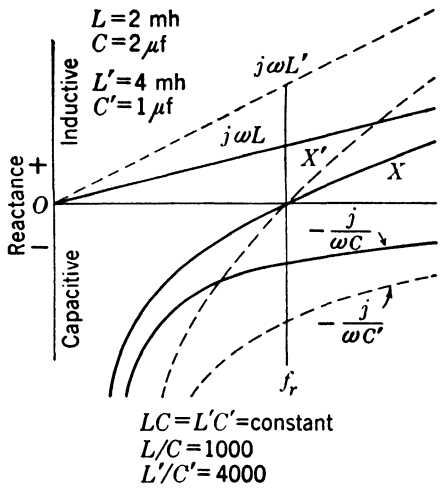


FIG. 12.8. Reactance curves showing change in L/C ratio.

Inductance-coil design usually fixes the Q of the coil, and if the resistance of the circuit joints is kept low, they will vary only slightly in comparison with the coil resistance. With a given value of Q, the selectivity will increase as L is made larger. The minimum capacity obtainable will be the limiting factor as to how large one can make this inductance and still tune the circuit to resonance at the desired frequency. This minimum capacity includes the distributed capacity of the coil plus the minimum capacity of the tuning capacitor. For instance, in the design of a tuned input circuit to a vacuum tube, the capacity of the leads and the grid-to-cathode capacity of the vacuum tube must be added to the minimum capacity of the tuning capacitor.

Example 12.6. If the minimum capacity across the input coil to a vacuum tube in a broadcast receiver is approximately 45 $\mu\mu$f, determine the inductance and the capacity range of a capacitor necessary to tune across the broadcast band from 550 to 1,600 kc.

Solution. By Eq. (12.5) at 1,600 kc

$$L = \frac{1}{4\pi^2 f^2 C} = \frac{1}{[(6.2832)(1.6 \times 10^6)]^2(45 \times 10^{-12})}$$
$$= \frac{1}{4{,}260} = 235\ \mu\text{h} \qquad \textit{Ans.}$$

Now at 550 kc the capacity by Eq. (12.5) is

$$C = \frac{1}{[(6.2832)(0.55 \times 10^6)]^2(235 \times 10^{-6})} = \frac{10^{-6}}{2{,}802}$$
$$= 357\ \mu\mu\text{f} = \text{maximum capacity} \qquad \textit{Ans.}$$

Example 12.7. Since 350-$\mu\mu$f capacitors are usually used to tune across the broadcast band, determine the minimum capacity and inductance necessary to use with the capacitor.

Solution. From Example 12.6, the ratio of $C_{max}/C_{min} = 357/45 = 7.84$, which means that the new minimum capacity will be

$$C_{min} = \frac{350}{7.84} = 44.6\ \mu\mu f = \text{minimum capacity} \qquad \textit{Ans.}$$

The new value of inductance by Eq. (12.5) is

$$L = \frac{1}{[(6.2832)(0.55 \times 10^6)]^2(350 \times 10^{-12})}$$
$$= 239.5\ \mu h \qquad \textit{Ans.}$$

For a broadcast receiver, it is not desirable to make the circuits too selective, for in doing so the required frequency band will be distorted. Even though, with careful design, the minimum capacity could be reduced considerably below 44.6 $\mu\mu$f with a resulting gain in selectivity, it may not be desirable if the receiver is to be used for broadcast reception.

12.6. Solution of Several Impedance Elements in Series. Many times it is desirable to find the total impedance of a series circuit containing several known elements. This can best be illustrated by the solution of some practical problems.

Example 12.8. Given a load with an impedance of $\dot{Z}_1 = R_1 + jX_1 = 70.7 + j70.7$ ohms, in series with an impedance of $\dot{Z}_2 = R_2 + jX_2 = 6 + j8$ ohms. If 110-volt alternating current is applied, determine the current flowing and the voltage across the load. Draw a voltage vector diagram.

Solution. The total impedance is obtained by adding $\dot{Z}_1$ and $\dot{Z}_2$ in rectangular form. (*Note:* Vectors are added in rectangular form and multiplied in polar form.)

$$\begin{aligned} \dot{Z}_1 &= 70.7 + j70.7 \\ \dot{Z}_2 &= 6 + j8 \\ \dot{Z} &= 76.7 + j78.7 = \text{total impedance} \end{aligned}$$

Changing this to polar form for use in Ohm's law gives

$$\dot{Z} = \sqrt{(76.7)^2 + (78.7)^2} \angle \tan^{-1}\frac{78.7}{76.7}$$
$$= 109.8\angle 45°44'$$

Taking the voltage as the reference vector, the current is

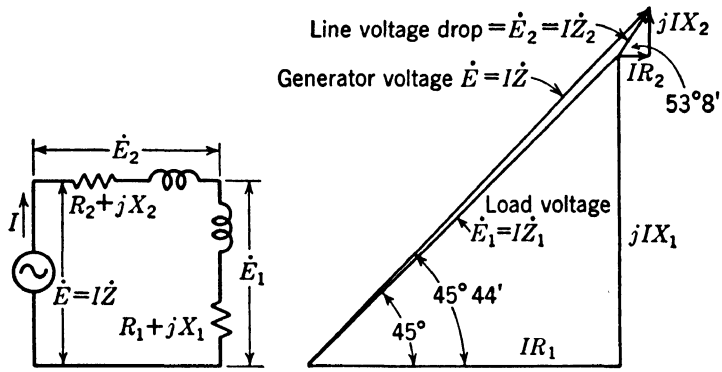
$$\dot{I} = \frac{\dot{E}}{\dot{Z}} \frac{110\angle 0°}{109.8\angle 45°44'} = 1.002\angle -45°44'\ \text{amp} \qquad \textit{Ans.}$$

which is the current flowing in the series circuit.

The voltage drop across the load is

$$\dot{E}_1 = \dot{I}\dot{Z}_1 = (1.002\angle 0°)(100\angle 45°)$$
$$= 100.2\angle 45°\ \text{volts} \qquad \textit{Ans.}$$

This example shows that the voltages across the various impedances add vectorially to give the resulting total voltage across the series circuit. This is a statement of *Kirchhoff's voltage law for an a-c circuit.* In other words, *the vectorial voltages around a series circuit form a closed polygon.*



(a) Circuit diagram (b) Voltage vector diagram

FIG. 12.9. Impedances in series.

In the preceding example, this polygon is a triangle made up of sides $\dot{E}$, $\dot{E}_1$, and $\dot{E}_2$. The voltages around this circuit can be divided up into the voltages across the various elements, such as IR_1, jIX_1, IR_2, jIX_2, and $I\dot{Z}$, forming a five-sided polygon as shown in Fig. 12.9.

Summary

Series resonance occurs in an a-c circuit when the inductive reactance is equal to the capacitive reactance. When this condition exists, the total impedance is a minimum and the current is a maximum. The frequency at which series resonance occurs is given by the relation $f_r = (1/2\pi)\sqrt{1/LC}$. In a series-resonant circuit the voltages appearing across the inductance and across the capacitance can exceed the total applied voltage.

The sharpness of resonance in a series resonant circuit depends upon the Q of the circuit. Q is defined as the ratio of inductive reactance to the resistance ($Q = 2\pi fL/R$) and is very nearly a constant, since both the inductive reactance and the resistance increase almost linearly with frequency. The sharpness of resonance, or the selectivity, of a resonant or tuned circuit depends also on the L/C ratio.

The Q of a series-resonant circuit can be determined from the resonance curve. On either side of resonance the power will be one-half the resonant power (the current will be 0.707 times the current at resonance) at frequencies f_1 and f_2 whose values depend on the Q of the circuit. They are related by the expressions $f_1 = f_r - (f_r/2Q)$ and $f_2 = f_r + (f_r/2Q)$. The difference $f_2 - f_1 = \Delta f$ is often called the *band width* of the circuit and is equivalent to $\Delta f = (1/Q)f_r$. At frequencies f_3 and f_4 where the power is reduced to 20 per cent of the resonant power (the current will be 44.7 per cent of the current at resonance) the width of the resonance curve is twice the band

width. These frequencies are related to the Q of the circuit by the expressions $f_3 = f_r - (f_r/Q)$ and

$$f_4 = f_r + \frac{f_r}{Q}$$

Therefore, knowing merely the Q and the resonant frequency of a series-resonant circuit, it is possible to obtain five points of the resonance curve: I_m at f_r, $0.707I_m$ at f_1 and f_2, $0.447I_m$ at f_3 and f_4.

Exercises

12.1. What one factor determines the characteristics of a series circuit? How do the resonance curves vary when this factor is changed? Explain.

12.2. A circuit is made up of an inductance of 300 μh, a resistance of 10 ohms, and a capacity of 300 $\mu\mu$f. What is the resonant frequency f_r? What is the impedance at this frequency? What is the phase relation of the voltage and the current?

12.3. When the power factor of the series circuit of Exercise 12.2 is 0.4462, what is the frequency? (Give two.)

12.4. If 250 volts is applied across the circuit of Exercise 12.2, what current will flow at resonance? When the power factor is 0.707? What voltage would be read across the inductance? The capacity?

12.5. Plot the resonance curves on graph paper for the series circuit of Exercise 12.2.

12.6. Plot the resonance curves for a circuit containing 600 μh inductance and 150 $\mu\mu$f capacity on the graph paper with the curves of Exercise 12.5. Have the two circuits the same resonant frequency? Why?

12.7. Explain the difference between LC product and L/C ratio and how they are used in the design of a resonant circuit.

12.8. Is it possible for the voltage to be higher across a piece of apparatus in a series circuit than the generator voltage across the series circuit? Why? Explain in detail.

12.9. A series circuit consists of three impedances

$$Z_1 = \underline{100/+60^\circ}$$
$$Z_2 = \underline{75/-30^\circ}$$
$$Z_3 = 50 - j50$$

connected across a generator of 500 volts. (*a*) Determine the total impedance of the circuit and draw a vector diagram to scale. (*b*) Calculate what current will flow, and give its phase relation to the applied voltage. (*c*) What will a voltmeter read if connected across Z_1, Z_2, and Z_3?

CHAPTER 13

ALTERNATING-CURRENT PARALLEL CIRCUITS

When two parts of a circuit are so connected that the same voltage is across both parts, they are said to be in *parallel.* Parallel circuits containing resistance only were discussed in Chaps. 3 and 4. In this chapter, capacity and inductance in parallel will be treated, with a discussion of the effect of resistance in series with them.

13.1. Parallel Resonance or Antiresonance Phenomena. The term *antiresonant* is used in connection with parallel circuits, because the line current reaches a minimum instead of a maximum as in the case of a series circuit at resonance. A parallel circuit is sometimes called a *tank circuit* because it acts like a tank, storing large circulating currents.

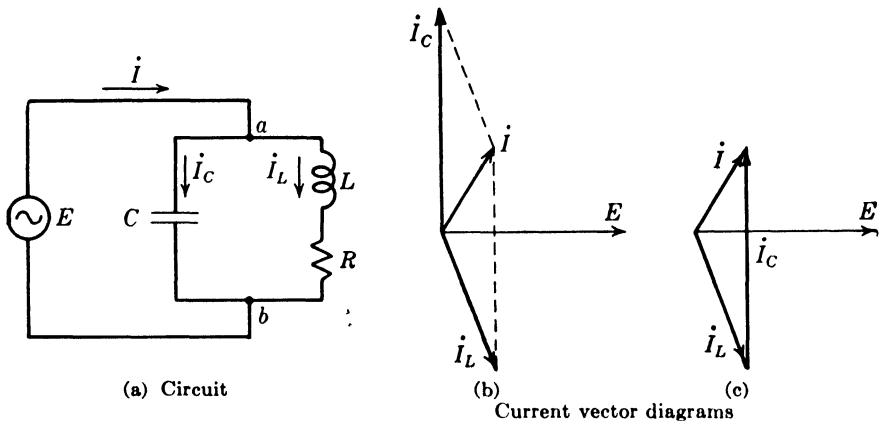


FIG. 13.1. Sine-wave voltage impressed on a parallel antiresonant circuit.

Consider the parallel antiresonant circuit of Fig. 13.1. This circuit consists of a capacitor and coil in parallel across a line. The resistance of the coil is lumped in R, and its inductance is lumped in L. In the inductive branch, the current will lag the voltage by nearly $-90°$ depending upon the amount of resistance in the coil. This is shown in Fig. 13.1*b*. If only the inductive branch were to be considered, I_L would normally be drawn along the positive x-axis, and then $\dot{E}$ would be at some positive angle, nearly $+90°$. However, since there is another parallel branch, the voltage $\dot{E}$ across this branch is taken as a reference vector and placed along the positive x-axis. Since the loss in capacitors is very small, it is

considered zero in this case; then the capacitive current I_C flowing through the capacitor leads the voltage across the parallel circuit by $+90°$. The currents $\dot{I}_L$ and $\dot{I}_C$ flow away from the point a, so the total current $\dot{I}$ flowing in the line must equal the sum of these currents, since *Kirchhoff's current law* must hold at this point. The current $\dot{I}$ in Fig. 13.1*b* is the vector sum of $\dot{I}_L$ and $\dot{I}_C$ (it is the diagonal of a parallelogram made up of sides $\dot{I}_L$ and $\dot{I}_C$). If the capacitive current $\dot{I}_C$ is moved parallel to itself to the other side of the parallelogram, the result will be a closed triangle. (*Note:* A *vector* can be moved without changing its value as long as its *magnitude* and *direction* are *not altered.*) The resulting triangle is shown in Fig. 13.1*c*. The equation for the currents at this junction is

$$\dot{I} = \dot{I}_L + \dot{I}_C \tag{13.1}$$

where $\dot{I}$ = total vector line current flowing to point a
$\dot{I}_1$ = vector current flowing through the inductive branch away from point a
$\dot{I}_C$ = vector current flowing through the capacitive branch away from point a

Since the admittance Y is proportional to the current I, the admittance vectors for this parallel antiresonant circuit can be written

$$\dot{Y} = \dot{Y}_L - \dot{Y}_C \tag{13.2}$$

where $\dot{Y}$ = total admittance vector of the parallel branch
$\dot{Y}_L$ = admittance vector of the inductive branch
$\dot{Y}_C$ = admittance vector of the capacitive branch

In terms of impedance vectors, this equation can be written

$$\frac{1}{\dot{Z}} = \frac{1}{\dot{Z}_L} + \frac{1}{\dot{Z}_C} = \frac{\dot{Z}_C + \dot{Z}_L}{\dot{Z}_L \dot{Z}_C} \tag{13.3}$$

or

$$\dot{Z} = \frac{\dot{Z}_L \dot{Z}_C}{\dot{Z}_L + \dot{Z}_C} \tag{13.4}$$

where $Y = \dfrac{1}{\dot{Z}}$

$$\dot{Y}_L = \frac{1}{\dot{Z}_L}$$

$$\dot{Y}_C = \frac{1}{\dot{Z}_C}$$

If this equation is applied to the parallel antiresonant circuit of Fig. 13.1*a*, the impedances of the parallel branches, if written in rectangular form, are

$$\dot{Z}_L = R + jX_L \tag{13.5}$$

$$\dot{Z}_C = -jX_C \tag{13.6}$$

Substituting these values in Eq. (13.4), and remembering that $j = \sqrt{-1}$, $j^2 = -1$, and $-j^2 = -(-1) = 1$, results in

$$\dot{Z} = \frac{(R + jX_L)(-jX_C)}{R + jX_L - jX_C} = \frac{X_L X_C - jRX_C}{R + j(X_L - X_C)} \tag{13.7}$$

This is a vector equation of the impedance $\dot{Z}$ but is not in a very convenient form, since both numerator and denominator are in rectangular form, that is, they both contain a resistive and reactive component. This equation can be rationalized by multiplying both numerator and denominator by the conjugate of the denominator.[1] This will remove the j term from the denominator, and the impedance can then be written in true rectangular form with a resistive and reactive component.

The conjugate of a complex number or vector is a *vector of equal magnitude but of opposite phase angle.* Hence when a number and its conjugate are multiplied together, the phase angles add to give zero. For example, let

$$\dot{c} = a + jb = c\angle\phi \tag{13.8}$$

then the conjugate of $\dot{c}$ is

$$a - jb = c\angle -\phi \tag{13.9}$$

And the product in rectangular form is

$$(a + jb)(a - jb) = a^2 + jab - jab - j^2b = a^2 + b^2 \tag{13.10}$$

Or the product in polar form is

$$c\angle\phi\, c\angle -\phi = c^2\angle\phi - \phi = c^2\angle 0^\circ = a^2 + b^2 \tag{13.11}$$

Since multiplying both numerator and denominator by the same thing does not change the value of the fraction, the conjugate of the denominator of Eq. (13.7) can be multiplied by both numerator and denominator as follows:

$$Z = \left[\frac{X_L X_C - jRX_C}{R + j(X_L - X_C)}\right]\left[\frac{R - j(X_L - X_C)}{R - j(X_L - X_C)}\right]$$

Performing the indicated multiplication gives

$$\dot{Z} = \frac{RX_L X_C - jR^2X_C - j(X_L - X_C)X_L X_C + j^2(X_L - X_C)RX_C}{R^2 + j(X_L - X_C) - j(X_L - X_C) - j^2(X_L - X_C)^2}$$

Expanding in the numerator and canceling terms in the denominator,

$$\dot{Z} = \frac{RX_L X_C - jR^2X_C - jX_L{}^2X_C + jX_L X_C{}^2 + j^2X_L X_C R - j^2X_C{}^2R}{R^2 + (X_L - X_C)^2}$$

[1] Carl E. Smith, *Applied Mathematics*, 1st ed., McGraw-Hill, New York, 1945, p. 98.

Canceling terms in the numerator,

$$\dot{Z} = \frac{RX_C^2 - jR^2X_C - jX_L^2X_C + jX_LX_C^2}{R^2 + (X_L - X_C)^2}$$

It will be noted that j appears in only three of the four terms in the numerator; hence, the fraction can be written as the sum of two fractions

$$\dot{Z} = \frac{RX_C^2}{R^2 + (X_L - X_C)^2} + j\frac{(X_LX_C^2 - X_L^2X_C - R^2X_C)}{R^2 + (X_L - X_C)^2} \quad (13.12)$$

This is the *rectangular form* for the general and exact equation of the impedance of a parallel circuit containing resistance and inductance in one branch and a pure capacity in the other branch.

Example 13.1. A parallel circuit as shown in Fig. 13.1*a* has

$X_L = 50$ ohms inductive reactance
$X_C = 75$ ohms capacitive reactance
$R = 5$ ohms resistance of the coil

Determine the impedance of the parallel circuit in polar and rectangular form.

Solution. Using Eq. (13.7)

$$\dot{Z} = \frac{X_LX_C - jRX_C}{R + j(X_L - X_C)} = \frac{(50)(75) - j(5)(75)}{5 + j(50 - 75)}$$

$$= \frac{3{,}750 - j375}{5 - j25} = \frac{3{,}769/-5°43'}{25.49/-78°41'}$$

$$= 147.8/72°58' \text{ ohms} \quad \text{(in polar form)} \quad \textit{Ans.}$$

$$= 147.8(0.29293 + j0.95613)$$

$$= 43.3 + j141.3 \text{ ohms} \quad \text{(in rectangular form)} \quad \textit{Ans.}$$

Substituting in Eq. (13.12)

$$\dot{Z} = \frac{(5)(75)^2}{5^2 + (50 - 75)^2} + j\frac{(50)(75)^2 - (50)^2(75) - (5)^2(75)}{5^2 + (50 - 75)^2}$$

$$= \frac{28{,}125}{650} + j\frac{281{,}250 - 187{,}500 - 1{,}875}{650}$$

$$= 43.3 + j141.3 = 147.8/72°58' \quad \textit{Ans.}$$

Example 13.1 shows that Eqs. (13.7) and (13.12) give exactly the same results. In Eq. (13.7), the denominator and numerator have to be changed from rectangular form to polar form in order to perform the division, and the answer then first appears in polar form, while Eq. (13.12 has been changed into rectangular form to begin with, hence the answer appears in rectangular form first. It then can be changed to polar form if necessary.

13.2. Resistance and Reactance Curves of a Parallel Antiresonant Circuit. Equation (13.12) is useful to obtain the antiresonant resistance and reactance curves of a parallel circuit containing resistance only in

the inductive branch. The first term gives the resistance component, and the second term gives the reactance component. These curves have been plotted in Fig. 13.2 for two values of resistance in the inductive branch. It will be noted that both resistance and reactance curves are sharper for the lower value of resistance. If this resistance were made zero, the resistance curve would go to positive infinity at antiresonance,

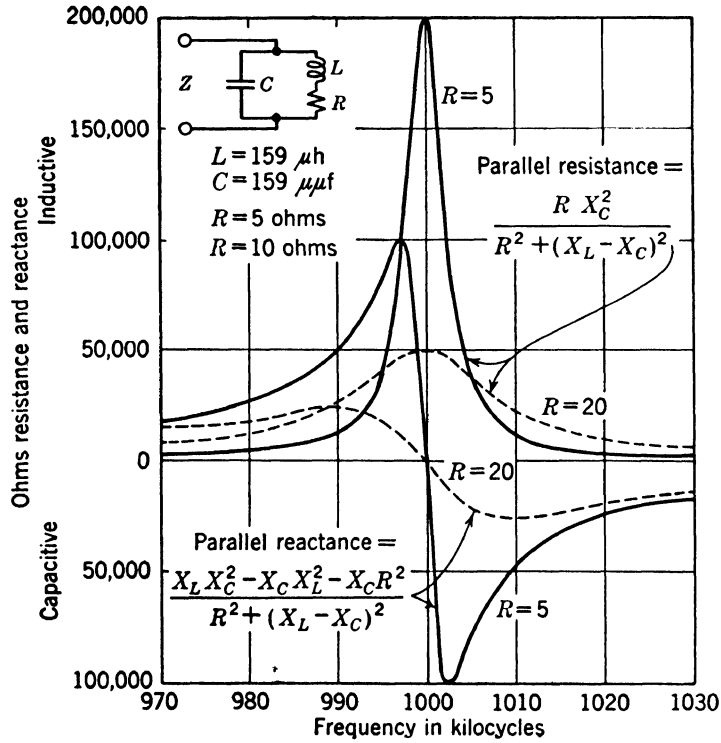


FIG. 13.2. Resistance and reactance curves of a parallel antiresonant circuit plotted as a function of frequency, with two values of resistance in the inductive branch.

and the reactance curve would jump from positive infinity to negative infinity at this same point.

13.3. Loci of Impedance and Admittance Vectors of a Parallel Antiresonant Circuit. The impedance of a parallel circuit is the vector sum of the resistance and reactance components as given in Eq. (13.12). Three such vectors have been plotted in Fig. 13.3. The terminus of the impedance vector traces out almost a circle as the frequency is increased and passes through antiresonance. The impedance vector Z_1 is for 997,500 cycles per sec. The impedance vector traces the second quadrant of the circle as the frequency is varied from 0 to 997,500. The first quadrant is traced as the frequency is varied from 997,500 to 1,000,000 (from

Z_1 to Z_2). Z_2 is the impedance value at the antiresonant frequency. The fourth quadrant is traced as the frequency varies from 1,000,000 to 1,002,500 (from Z_2 to Z_3), and the third quadrant is traced as the frequency increases from 1,002,500 to negative infinity. From this it is seen that half the circle is traced, while the frequency varies from 997,500 to 1,002,500 or from vector Z_1 to vector Z_3. Since the terminus of the impedance vector traces *almost* a circle, its reciprocal, or the admittance of a parallel circuit, will trace *almost* a straight line. This is opposite to the effect in a series circuit as shown in Fig. 12.4.

The impedance curves have practically the same shape as the resistance curves, as will be seen if the resistance curves of Fig. 13.2 are compared with the impedance curves of Fig. 13.8. The impedance curves for the L/C ratio are plotted for 5 and 20 ohms and are the vector sums of the resistance and reactance curves of Fig. 13.2. The resistance curves are just slightly sharper than the impedance curves, owing to the added effect of the reactance in the circuit. At antiresonance, the reactance is zero; hence the peaks of the resistance and impedance curves are equal.

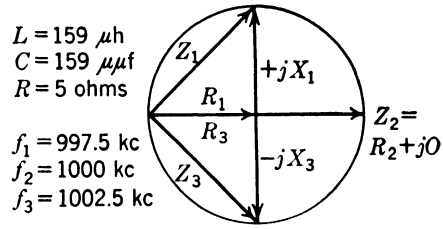


FIG. 13.3. Locus of impedance vectors in a parallel circuit as the frequency passes through antiresonance (almost a circle).

13.4. Definition of Conductance and Susceptance Vectors. In parallel circuits it is easier to add admittances than to add the reciprocals of impedances, just as in the case of pure resistive circuits it is easier to add conductances than to add the reciprocals of resistances. When the parallel circuits contain both resistance and reactance, it becomes convenient to define the in-phase component of the admittance as the conductance and the out-of-phase component as the susceptance.

In general, consider the reciprocal of the vector $\dot{c}$ as given in Eq. (13.8), and then rationalize the fraction by multiplying both numerator and denominator by the conjugate of the denominator; thus

$$\frac{1}{\dot{c}} = \left(\frac{1}{a + jb}\right)\left(\frac{a - jb}{a - jb}\right) = \frac{a - jb}{a^2 + b^2}$$
$$= \frac{a}{a^2 + b^2} - j\frac{b}{a^2 + b^2} \tag{13.13}$$

If $\dot{c}$ is an impedance vector, then its reciprocal is called an *admittance vector*, and the real and imaginary portions of this vector as defined in Eq. (13.13) are called the *conductance* and *susceptance*, respectively, of the admittance vector.

The vector $\dot{c}$ and its reciprocal are shown in Fig. 13.4. It will be noted that the vector $\dot{c} = a + jb$ lies above the reference axis by an angle ϕ, and the reciprocal lies below this axis by the same angle measured as $-\phi$.

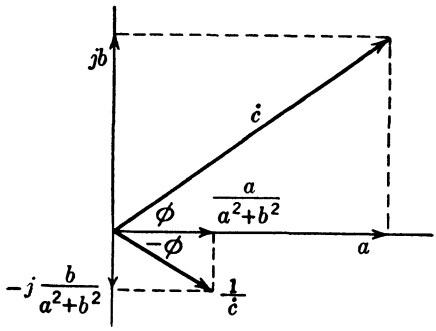


FIG. 13.4. Reciprocal of a complex number in rectangular or polar form.

The admittance of the inductive branch of the parallel circuit shown in Fig. 13.1*a* can be written as the reciprocal of the impedance and then rationalized as was done in Eq. (13.13) to obtain

$$\dot{Y}_L = \frac{1}{\dot{Z}_L} = \frac{1}{R + jX_L} = \frac{R}{R^2 + X_L^2} - j\frac{X_L}{R^2 + X_L^2} = G - jB_L \quad (13.14)$$

where $$G = \frac{R}{R^2 + X_L^2} = \text{conductance, mhos} \quad (13.15)$$

and $$B_L = \frac{X_L}{R^2 + X_L^2} = \text{susceptance, mhos} \quad (13.16)$$

The last two equations can be taken as the definitions of conductance and susceptance, respectively, in terms of the resistance and reactance of the circuit. Similarly, the resistance and reactance of a circuit can be defined in terms of the conductance and susceptance of the circuit; thus

$$\dot{Z}_L = \frac{1}{\dot{Y}_L} = \frac{1}{G - jB_L} = \frac{G}{G^2 + B_L^2} + j\frac{B_L}{G^2 + B_L^2} = R + jX_L \quad (13.17)$$

where $$R = \frac{G}{G^2 + B_L^2} = \text{resistance, ohms} \quad (13.18)$$

and $$X_L = \frac{B_L}{G^2 + B_L^2} = \text{reactance, ohms} \quad (13.19)$$

Example 13.2. Solve Example 13.1 by the application of conductance and susceptance vectors.

Solution. By Eqs. (13.15) and (13.16) the conductance and susceptance of the inductive branch are

$$G = \frac{5}{5^2 + 50^2} = 0.00198 \text{ mho}$$

$$B_L = \frac{50}{5^2 + 50^2} = 0.01980 \text{ mho}$$

The admittance of this branch is then, by Eq. (13.14),

$$\dot{Y}_L = G - jB_L = 0.00198 - j0.0198$$

Since the resistance of the capacitive branch is zero, Eqs. (13.14) and (13.15) reduce to

$$G = 0$$

and

$$B_C = \frac{1}{X_C} = \frac{1}{75} = 0.01333 \text{ mho}$$

The admittance of this branch is then

$$\dot{Y}_C = 0 + jB_C = j0.01333 \text{ mho}$$

Now by Eq. (13.2) the total admittance is

$$\dot{Y} = \dot{Y}_L + \dot{Y}_C = G - jB_L + jB_C = 0.00198 - j0.0198 + j0.01333$$
$$= 0.00198 - j0.00647 = 0.006766/\underline{-72°58'}$$

And the corresponding impedance is

$$\dot{Z} = \frac{1}{\dot{Y}} = \frac{1}{0.006766/\underline{-72°58'}} = 147.8/\underline{72°58'}$$
$$= 43.3 + j141.3 \qquad \textit{Ans.}$$

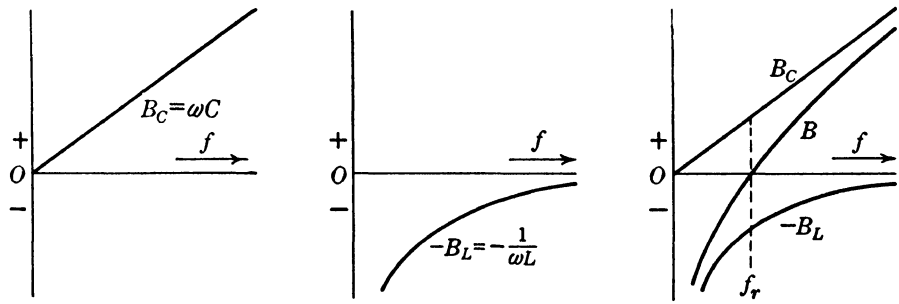
It will be noted that this method gives the same answer as that obtained in Example 13.1.

13.5. Susceptance Curves of a Parallel Circuit. Consider the theoretical case of a *parallel circuit lacking resistance* in *either branch.* Then the susceptance is the reciprocal of the reactance. For instance, the susceptance B_C of the capacitive branch is

$$jB_C = j\omega C = \frac{j}{1/\omega C} = \frac{j}{X_C} = \frac{1}{-jX_C} \tag{13.20}$$

and the susceptance B_L of the inductive branch is

$$-jB_L = \frac{-j}{\omega L} = \frac{1}{jX_L} \tag{13.21}$$



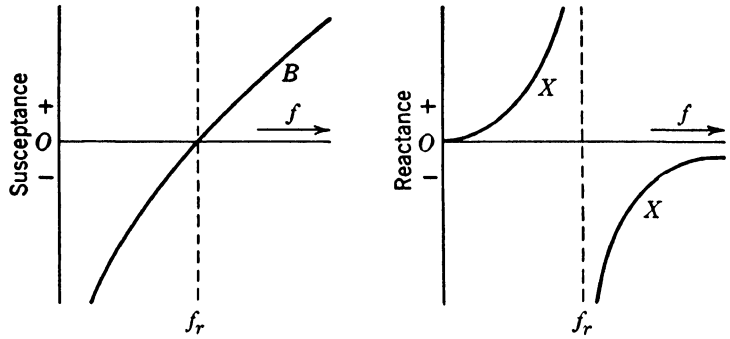
(a) Capacitive susceptance (b) Inductive susceptance (c) Total susceptance

FIG. 13.5. Susceptance curves of a parallel antiresonant circuit.

Figure 13.5*a* shows how the susceptance of a capacitor varies with frequency; Fig. 13.5*b* shows how the susceptance of an inductance varies with frequency, and Fig. 13.5*c* shows how the two curves can be added to

obtain the total susceptance of a parallel circuit. It will be noted that these curves are exactly similar to the reactance curves of a series circuit as shown in the last chapter.

The negative *reciprocal* of the *total-susceptance curve* is a *total-reactance curve.* The total-susceptance curve B of Fig. 13.5*c* is redrawn in Fig. 13.6*a*, and its reciprocal is shown in Fig. 13.6*b*. An explanation of this



(a) Total susceptance (b) Total reactance

FIG. 13.6. Susceptance and reactance curves for a parallel antiresonant circuit containing no resistance.

reciprocal relation can be obtained from Eqs. (13.20) and (13.21). In terms of total susceptance and reactance, these equations are

$$+jB = \frac{1}{-jX} \qquad \text{and} \qquad -jB = \frac{1}{jX} \tag{13.22}$$

For example, take a point on the susceptance curve B of Fig. 13.6*a* below the x-axis. If $-jB$ is large and the frequency is low, its reciprocal will be a small positive quantity as shown in the total reactance curve of Fig. 13.6*b*. Zero reactance corresponds to an infinite negative susceptance. As the frequency increases to the antiresonant frequency f_r, the susceptance B becomes a small negative number, and its negative reciprocal will be a large positive number, until at zero susceptance it becomes an infinite positive reactance. As the susceptance becomes positive after passing through zero (as it crosses the x-axis), its reciprocal jumps from positive infinity to negative infinity and then becomes a smaller negative reactance as the frequency is increased and the susceptance becomes a larger positive quantity.

The reactance curves of Fig. 13.6*b* are for pure reactive elements in parallel. In actual practice, there is always some resistance, and the result is that these reactance curves change into the form shown in Fig. 13.2. They do not go to infinity but reach, as will be shown later, a maximum at a fractional deviation of $1/2Q$ of the resonant frequency.

13.6. Optimum L/C Ratio. An inspection of the susceptance curves of a parallel circuit shows how the susceptance of a parallel circuit depends upon the L/C ratio. Figure 13.7 shows the susceptance curves for a parallel circuit for two values of the L/C ratio. In each case, the LC product is kept the same.

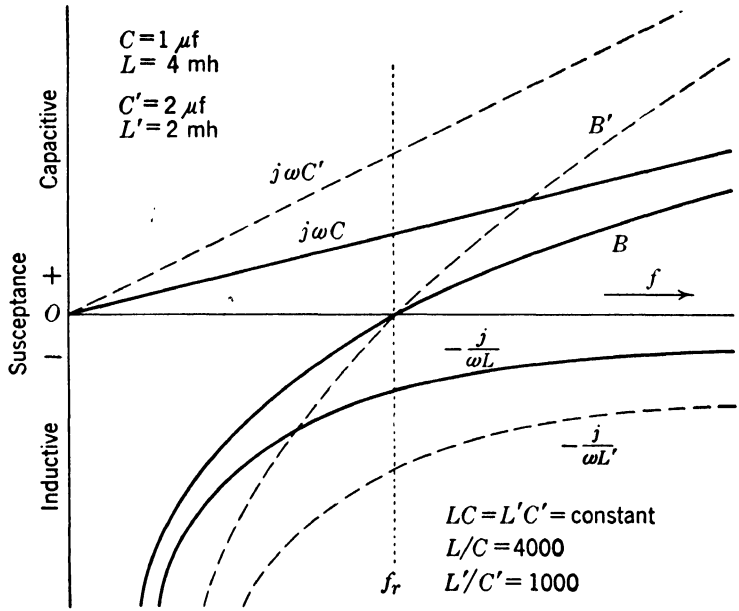


FIG. 13.7. Susceptance curves, showing change in L/C ratio.

The curves in Fig. 13.7 are susceptance curves; therefore parallel-circuit selectivity is increased by increasing the slope of the curve. The magnitude of this improvement in selectivity is shown in Fig. 13.8. For a parallel circuit with 5 ohms resistance, the circuit impedance at resonance is increased from 50,000 to 200,000 ohms by multiplying the L/C ratio by 4. This means that the slope of the sides of the antiresonance curve is much more for the case of the higher L/C ratio.

The *Q of the circuit* tells, more than any other one factor, the *merit of the circuit.* To illustrate this important point, an example has been solved for two L/C ratios and two values of resistance in the inductive branch for each case and have been plotted in Fig. 13.8. These curves clearly show that the sharpness of the antiresonant impedance curves is decreased when the L/C ratio is decreased. The impedance of the circuit at resonance is proportional to the L/C ratio.

Even though the resistance of the parallel antiresonant circuit will increase when the inductance is increased, owing to the increased number of turns, the resistance increases only directly as the number of turns,

while inductance increases as the square of the number of turns. Hence, the Q of the circuit will be improved if the L/C ratio is increased. As in the case of the series circuit, usually the mechanical design of the variable capacitor will determine the capacity range and hence the L/C ratio.

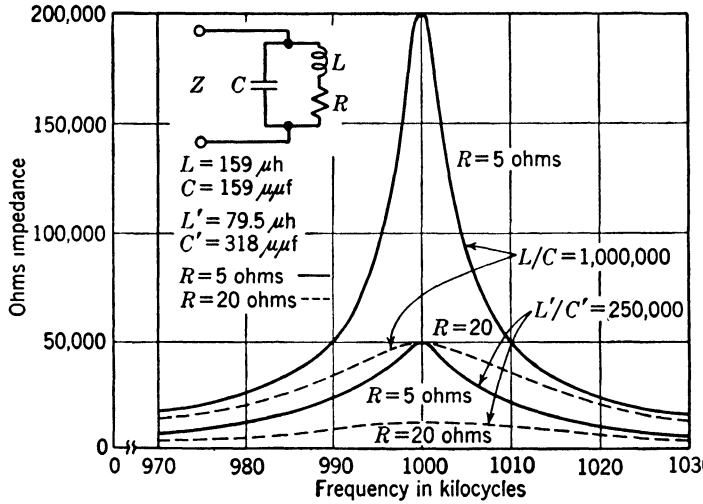


FIG. 13.8. Impedance of a parallel antiresonant circuit for two values of resistance in the inductive branch and for two L/C ratios.

If more selectivity is desired, resort must be made to decreasing the coil resistance by improved coil design; by this means, the Q of the circuit will be increased.

13.7. Approximate Equations of Impedance of a Parallel Antiresonant Circuit. *a. Resistance in Inductance Branch.* If the resistance of the coil is low, the condition for antiresonance is that the inductive reactance is equal to the capacitive reactance, that is, $X_L = X_C$. This is the same condition as that for resonance in a series circuit containing L and C. Making this substitution in Eq. (13.7) gives

$$Z = \frac{X_L X_C - jRX_C}{R} \tag{13.23}$$

Since R is very small in comparison with X_C, the second term in the numerator can be neglected in comparison with the product $X_L X_C$, and the *approximate equation* can be written

$$Z \doteq \frac{X_L X_C}{R} = \frac{X_L^2}{R} = X_L Q = \frac{X_C^2}{R} = X_C Q \tag{13.24}$$

Substituting $X_L = 2\pi fL$ and $X_C = \frac{1}{2\pi fC}$ in the above equation gives

$$Z \doteq \frac{X_L X_C}{R} = \frac{2\pi fL}{R 2\pi fC} = \frac{L}{RC} \tag{13.25}$$

Equations (13.24) and (13.25) are extremely important practical equations for the condition of parallel antiresonance where the resistance R is relatively small in comparison with the inductive reactance. These equations should be *memorized.*

Example 13.3. Given a parallel antiresonant circuit with the following constants:

$$C = 169\ \mu\mu\text{f}, \qquad f = 1{,}000\ \text{kc}$$
$$L = 150\ \mu\text{h}, \qquad Q = 100$$

Determine the impedance from the various approximate equations.

Solution

$$X_L = 2\pi fL$$
$$= (6.2832)(10^6)(150 \times 10^{-6})$$
$$= 942 \text{ ohms}$$

$$R = \frac{X_L}{Q} = \frac{942}{100} = 9.42 \text{ ohms}$$

$$X_C = \frac{1}{2\pi fC} = \frac{1}{(6.2832)(10^6)(169 \times 10^{-12})}$$
$$= 942 \text{ ohms}$$

$$Z \doteq \frac{X_L X_C}{R} = \frac{(942)(942)}{9.42} = 94{,}200 \text{ ohms} \qquad \textit{Ans.}$$

$$\doteq \frac{X_L^2}{R} = \frac{(942)^2}{9.42} = 94{,}200 \text{ ohms} \qquad \textit{Ans.}$$

$$\doteq X_L Q = (942)(100) = 94{,}200 \text{ ohms} \qquad \textit{Ans.}$$

$$\doteq \frac{X_C^2}{R} = \frac{(942)^2}{9.42} = 94{,}200 \text{ ohms} \qquad \textit{Ans.}$$

$$\doteq X_C Q = (942)(100) = 94{,}200 \text{ ohms} \qquad \textit{Ans.}$$

$$\doteq \frac{L}{RC} = \frac{150 \times 10^{-6}}{(9.42)(169 \times 10^{-12})} = 94{,}200 \text{ ohms} \qquad \textit{Ans.}$$

b. Resistance in Both Parallel Branches. If both branches of the parallel antiresonant circuit contain resistance as shown in Fig. 13.9, the *approximate equations* (13.24) and (13.25) can be written with

$$R_t = R_L + R_C \qquad (13.26)$$

Thus the *approximate equation* for the impedance is

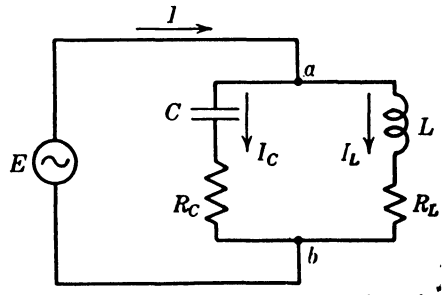


FIG. 13.9. Resistance in both branches of a parallel antiresonant circuit.

$$Z \doteq \frac{X_L X_C}{R_L + R_C} = \frac{X_L^2}{R_L + R_C} = X_L Q_t = \frac{X_C^2}{R_L + R_C} = X_C Q_t$$
$$= \frac{L}{C(R_L + R_C)} \qquad (13.27)$$

where

$$Q_t = \frac{X_L}{R_L + R_C} = \frac{X_C}{R_L + R_C} \qquad (13.28)$$

This approximate equation is accurate only when $R_L + R_C$ is small in comparison with X_L and X_C.

13.8. Approximate Resonance Curves of a Parallel Antiresonant Circuit. A careful examination of Eq. (13.4) reveals that the numerator $\dot{Z}_L\dot{Z}_C$ is approximately a constant if the Q is reasonably high, and that the denominator $\dot{Z}_L + \dot{Z}_C$ is the series impedance of the circuit; hence the impedance curves of a parallel circuit will have almost exactly the same shape as the admittance curves of a series circuit.

At antiresonance, the voltage across the parallel circuit can, by Ohm's law and Eq. (13.24), be written

$$E_r = IZ = \frac{IX_L^2}{R} = IX_LQ = IRQ^2 \qquad \text{(at antiresonance)} \quad (13.29)$$

The current in the inductance branch at antiresonance is

$$I_L = \frac{E}{X_L} = IQ \qquad (13.30)$$

Since $X_L = X_C$, the current in the capacitive branch is also

$$I_C = \frac{E}{X_C} = IQ \qquad (13.31)$$

These equations show a resonant rise of Q times the applied line current. Equations (13.30) and (13.31) are for a parallel circuit, as Eqs. (12.17) and (12.18) are for a series circuit.

Five points on the resonance curves of a parallel circuit can be obtained from Eqs. (12.6), (12.30), (12.31), (12.32), and (12.33). In Fig. 12.6, the reactance curves are replaced by susceptance curves, and the resonance curves of a parallel circuit are curves of the voltage or impedance instead of current or admittance as in a series circuit. The line voltage leads the line current for all frequencies below antiresonance, just as the line current leads the line voltage for all frequencies below resonance in a series circuit. Similarly, above antiresonance the line voltage lags the line current in an antiresonant circuit, while the line current lags the line voltage in a series-resonant circuit.

Example 13.4. A parallel circuit with resistance only in the inductive branch has the following constants:

$$L = 159\ \mu\text{h}, \qquad R = 5\ \text{ohms}$$
$$C = 159\ \mu\mu\text{f}, \qquad I = 10\ \text{ma}$$

Determine the Q of the parallel circuit at antiresonance, the inductive and capacitive branch currents at antiresonance, the frequency discrimination, and five points, with their associated phase angles, to give the approximate resonant curve of the voltage.

Solution

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{6.2832\sqrt{(159 \times 10^{-6})(159 \times 10^{-12})}}$$

$= 1{,}000{,}000$ cycles/sec *Ans.*

$$Q = \frac{\omega L}{R} = \frac{(6.2832 \times 10^6)(159 \times 10^{-6})}{5} = 200 \quad \textit{Ans.}$$

$$I_L = I_C = IQ = (0.01)(200) = 2 \text{ amp} \quad \textit{Ans.}$$

$$\frac{f_2 - f_1}{f_r} = \frac{1}{Q} = \frac{1}{200} = 0.005 \quad \textit{Ans.}$$

or $$\frac{100}{Q} = \frac{100}{200} = 0.5\% = \text{frequency discrimination}$$

$$f_1 = 1{,}000{,}000 - \frac{1{,}000{,}000}{(2)(200)} = 997{,}500 \text{ cycles/sec} \quad \textit{Ans.}$$

$$f_2 = 1{,}000{,}000 + 2{,}500 = 1{,}002{,}500 \text{ cycles/sec} \quad \textit{Ans.}$$

$$f_3 = 1{,}000{,}000 - \frac{1{,}000{,}000}{200} = 995{,}000 \text{ cycles/sec} \quad \textit{Ans.}$$

$$f_4 = 1{,}000{,}000 + 5{,}000 = 1{,}005{,}000 \text{ cycles/sec} \quad \textit{Ans.}$$

The voltages corresponding to these frequencies, taken with respect to the resonant line current, are

$$\dot{E}_r = IRQ^2 = (0.01)(5)(200^2)\angle 0° = 2000\angle 0° \text{ volts} \quad \textit{Ans.}$$

$$\dot{E}_1 = (2{,}000)(0.707)\angle +45° = 1{,}414\angle +45° \quad (\text{at } f_1) \quad \textit{Ans.}$$

$$\dot{E}_2 = (2{,}000)(0.707)\angle -45° = 1{,}414\angle -45° \quad (\text{at } f_2) \quad \textit{Ans.}$$

$$\dot{E}_3 = (2{,}000)(0.447)\angle +63.5° = 894\angle +63.5° \quad (\text{at } f_3) \quad \textit{Ans.}$$

$$\dot{E}_4 = (2{,}000)(0.447)\angle -63.5° = 894\angle -63.5° \quad (\text{at } f_4) \quad \textit{Ans.}$$

A striking parallel will be noted between this example and Example 12.4. In this example the line current I was held constant, while in the series circuit the generator voltage was held constant.

13.9. Special Case for $R = \sqrt{L/C}$ in Both Branches of a Parallel Antiresonant Circuit. If in Fig. 13.9 the resistances in both branches are made equal, say, let $R = R_L = R_C$, then the impedance of the parallel circuit can be written

$$Z_L = R + jX_L \tag{13.32}$$

$$Z_C = R - jX_C \tag{13.33}$$

Substituting these values in Eq. (13.4) gives

$$\dot{Z} = \frac{(R + jX_L)(R - jX_C)}{2R + jX_L - jX_C}$$

$$= \frac{R^2 + X_L X_C + jR(X_L - X_C)}{2R + j(X_L - X_C)} \tag{13.34}$$

When $R = \sqrt{L/C}$, we obtain an interesting result from Eq. (13.34). Substituting for the product

$$X_L X_C = \frac{2\pi fL}{2\pi fC} = \frac{L}{C} = R^2 \tag{13.35}$$

in Eq. (13.34) gives

$$\dot{Z} = \frac{2R^2 + jR(X_L - X_C)}{2R + j(X_L - X_C)} = R = \sqrt{\frac{L}{C}} \tag{13.36}$$

This equation shows that the equivalent impedance of the parallel combination is *independent* of *frequency for all frequencies;* hence the circuit is antiresonant at all frequencies, and the voltage is at all times in phase with the current. Such a circuit is of value to match the surge impedance of communication lines, which have a surge impedance of $\sqrt{L/C}$. Since this type of load impedance stays constant as the frequency is varied, it finds practical applications in wide-band television video circuits. Owing to the large amount of resistance, however, the circuits are usually rather low in efficiency.

13.10. Comparisons of Series and Parallel Circuits. 1. At resonance the impedances of both series and parallel circuits are resistive.

2. At resonance the series circuit offers a minimum impedance, while the parallel circuit offers a maximum impedance.

3. At resonance the impedance of a series circuit is increased by adding resistance, while the impedance of a parallel circuit is decreased when a small resistance is increased slightly in one or both of the parallel branches.

4. At frequencies below resonance, a series circuit has a capacitive reactance, while a parallel circuit has an inductive reactance.

5. At frequencies below resonance, the current in a series circuit will lead the voltage, while the current in the line feeding a parallel circuit will lag the voltage.

6. At frequencies above resonance, a series circuit has an inductive reactance, while a parallel circuit has a capacitive reactance.

7. At frequencies above resonance, the current in a series circuit will lag the voltage, while the current in the line feeding a parallel circuit will lead the voltage.

8. The resonance curves of a series circuit are made sharper by decreasing the resistance of the circuit.

9. The antiresonance curves of a parallel circuit are made sharper by decreasing the resistance of the circuit.

10. The slope of the total-reactance curve in a series circuit is increased when the L/C ratio is increased.

11. The slope of the total-susceptance curve in a parallel circuit is increased when the L/C ratio is decreased.

Note: These comparisons are for resonance in a series circuit and antiresonance in a parallel circuit.

Series- and parallel-resonance phenomena are perhaps the most *fundamental of all ideas in the analysis of a-c networks.*

Summary

To understand the performance of parallel-resonant circuits, the student will find a comparison with series-resonant circuits is useful. Consider at the same time the impedance for both series and parallel circuits.

$$\dot{Z}_s = R + j(X_L - X_C)$$

$$\dot{Z}_p = \frac{RX_C^2}{R^2 + (X_L - X_C)^2} + j\frac{(X_L X_C^2 - X_L^2 X_C - R^2 X_C)}{R^2 + (X_L - X_C)^2}$$

At resonance, the series circuit has an impedance $Z_s = R$, which is usually low. At resonance, the parallel circuit has an impedance

$$Z_p = \frac{RX_C^2}{R^2 + (X_L - X_C)^2}$$

which for reasonably high Q circuits reduces to

$$Z_p \doteq \frac{X_C^2}{R} = \frac{X_L^2}{R} = QX_L = QX_C$$

resulting in a high impedance. This can also be written $Z_p \doteq L/CR$ by substituting $\omega L(1/\omega C)$ for $(\omega L)^2 = X_L^2$.

From the preceding relationships, it is evident that at resonance, both the series and parallel circuits have impedances that are resistive. (This is true for parallel resonance only when the Q is high or when resonance is taken as the condition for unity power factor.)

It is also quite obvious that at resonance the series circuit will have a minimum impedance, while the parallel circuit will have a maximum impedance (again only if Q is large).

The two equations for the impedance readily show the effect of added resistance on the impedance at resonance. In the series circuit, the impedance Z_s, at the resonant frequency, is directly proportional to (actually equal to) the resistance, and increasing the circuit resistance will increase the impedance and reduce the current at resonance. In the parallel circuit the impedance Z_p at resonance is inversely proportional to the resistance, and increasing the resistance will reduce the impedance and increase the current at resonance.

At frequencies below resonance, the inductive reactance X_L is less than the capacitive reactance X_C. Therefore, in the series circuit, at frequencies below the resonant frequency, the total net reactance is capacitive, being merely the difference between the two reactances. In the parallel circuit, since the inductive reactance is lower, the current through this branch will be greater and the line current will be increased, with the inductive component greater than the capacitive component. Hence, at frequencies below resonance, the line current will be inductive and the reactance will be inductive for the parallel circuit.

From the previous statements, it is evident that below resonance the current in the series circuit will lead the voltage, since the reactance is capacitive. It is also obvious that for the parallel circuit, the current will lag the voltage at frequencies below the resonant frequency since the reactance is inductive.

At frequencies above resonance, the opposite conditions will exist in both types of circuits to those existing below resonance.

In both series- and parallel-resonant circuits, increasing the L/C ratio and increasing the Q increases the selectivity.

Parallel resonance or antiresonance occurs when the line current reaches a minimum, when the power factor is unity, or when the inductive reactance is equal to the capacitive reactance.

The presence of appreciable resistance in the branches of a parallel-resonant circuit affects the frequency of resonance as well as the selectivity.

The impedance at resonance of a parallel-resonant circuit is equal to Q times the inductive reactance. Hence, parallel resonance results in a resonant rise of impedance of Q times that of either the inductance or capacitance.

Exercises

13.1. How does the Q of a parallel circuit affect the reactance, resistance, and impedance curves?

13.2. Given a tank circuit with

$$L = 75\ \mu\text{h}$$
$$C = 338\ \mu\mu\text{f}$$
$$R = 10\ \text{ohms}$$

Determine (*a*) the antiresonance frequency, (*b*) the maximum inductive and capacitive reactance, and (*c*) the frequency that will give maximum inductive and capacitive reactance.

13.3. Find the approximate impedance of the parallel circuit of Exercise 13.2 at antiresonance. If the resistance $R = 5$ ohms, what will be the approximate impedance of the circuit to a signal of 1 Mc?

13.4. If an inductance of 150 μh is available, determine the capacity necessary to tune it to 1 Mc to make a wave trap. If the inductance has 10 ohms, will it be a better wave trap at this frequency than the circuit of Exercise 13.2?

13.5. Explain why and how a meter in the line to a parallel circuit can be used to determine antiresonance.

13.6. When is a parallel circuit resistive, inductive, and capacitive? Explain why these conditions are true.

13.7. If $L = 1.25$ mh and $C = 2.5$ μf, determine the resistance that will be necessary in each parallel branch to make the circuit resistive at all frequencies. What will be the impedance of this circuit?

13.8. Explain the value of susceptance curves in detail. How does the total susceptance vary as the frequency passes through antiresonance?

13.9. How does the current in the line and two branches of a parallel circuit vary as the frequency passes through antiresonance? Illustrate with a vector diagram.

13.10. Draw a vector diagram for $\dot{I}_C$ and $\dot{I}_L$ of Exercise 13.7 when the frequency $f_1 = 1{,}643$ cycles/sec, $f_2 = 2{,}850$ cycles/sec, and $f_3 = 4{,}934$ cycles/sec. Do the terminal points of these current vectors traverse a circle? Use the line voltage as the reference vector. From the vector diagrams, explain why the impedance is always pure resistance.

CHAPTER 14

SERIES-PARALLEL ALTERNATING-CURRENT CIRCUITS

In the last two chapters, series and parallel circuits have been treated. *Series circuits* are *resonant* at one frequency, while *parallel circuits* are *antiresonant* at one frequency. A circuit may have both resonant and antiresonant properties. Such circuits must have more than one mesh to have more than one of these properties. Furthermore, owing to the positive slope of the reactance curves in both series and parallel circuits, the number of resonant frequencies cannot differ from the number of antiresonant frequencies by more than one, because the curve must cross the x-axis for resonance and go to positive infinity for antiresonance before it can jump to negative infinity and again cross the x-axis for resonance. However, if resistance is present in the circuits, the antiresonance curves will not go to positive infinity and negative infinity but will reach a maximum on either side of antiresonance, as was discussed in the last chapter.

14.1. Series-parallel A-c Circuits. One of the most common historical applications of series, parallel, and series-parallel circuits is the *wave trap*, which utilizes the principle of resonance and antiresonance in the elimination of undesired signals. The following treatment of wave traps is only qualitative, showing roughly how one signal may be discriminated against another signal.

The tank circuit of Fig. 14.1*a*, when tuned to the undesired frequency, offers a very high impedance; hence most of the undesired voltage drop will be across the parallel antiresonant circuit and very little of the undesired signal voltage will be across the receiver input terminals. The series circuit of Fig. 14.1*b*, when tuned to the undesired frequency, will offer zero impedance to ground, hence will be shorted out, with the result that none of the undesired signal voltage will be across the receiver input terminals.

The tank circuit of Fig. 14.1*c*, when tuned to the desired frequency, will offer a high impedance with the result that most of the voltage will be across the parallel antiresonant circuit, hence across the receiver input terminals.

The series circuit of Fig. 14.1*d*, when tuned to the desired frequency, will offer minimum impedance, and hence will carry maximum current, which will cause maximum voltages across either the inductance or the

capacity. The receiver is connected across the capacity to utilize this desired voltage.

The series-parallel circuit of Fig. 14.1*e* has two adjustments. The upper capacitor across the inductance is tuned to make this circuit anti-resonant to the undesired signal; hence it will have a high impedance,

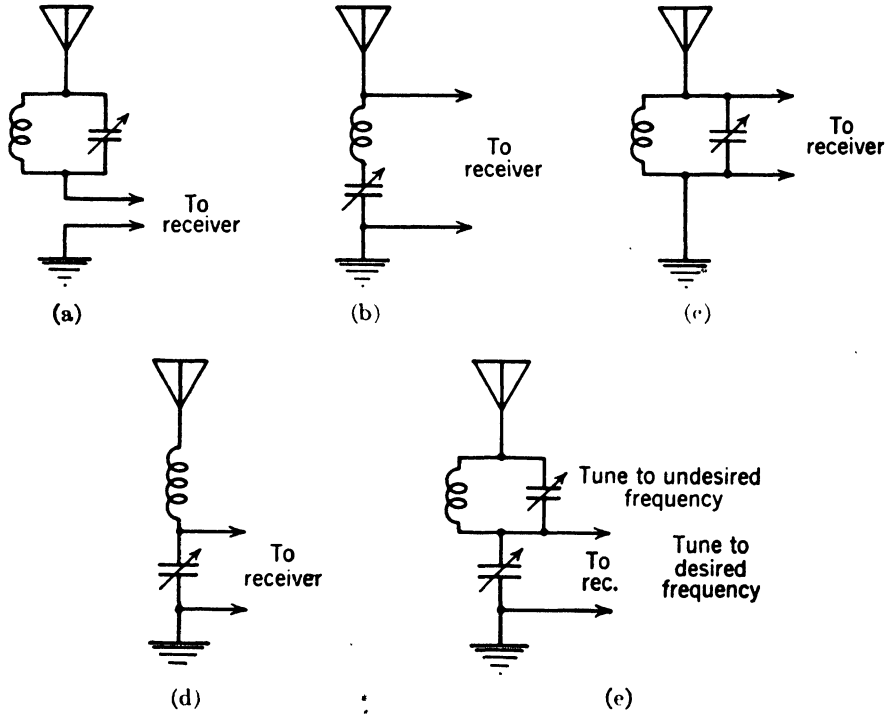


FIG. 14.1. Some typical series parallel, and series-parallel wave trap circuits.

and most of the voltage drop will be across this element. The capacitor below is tuned to give series resonance with the coil for the desired frequency. This will cause a high current of the desired frequency to flow, and hence a high voltage across the lower capacitor, which is fed to the receiver.

14.2. Reactance Sketch for a Series-parallel Circuit. Already reactance sketches have been made for series and parallel circuits alone. A series-parallel circuit with the reactance sketch is shown in Fig. 14.2*a*. Remembering that susceptance curves must be used to add the effect of elements in parallel, it may be seen in Fig. 14. 2*b* how the susceptance of the capacitor, B_{C1}, is added to the susceptance of the inductance, B_{L1}, to obtain their sum B_1. Changing the total susceptance B_1 of the parallel circuit to X_1, the negative reciprocal is taken, giving the X_1 reactance

curve of Fig. 14.2*c*. The total reactance X_1 of the parallel circuit can now be added to the reactance X_{C2} of the capacitor that is in series. Figure 14.2*d* shows the graphical addition of X_1 and X_{C2}. The total reactance, which is dotted in Fig. 14.2*d*, is drawn solid in Fig. 14.2*e*. The curve of Fig. 14.2*e* is the total-reactance curve of the series-parallel circuit of Fig. 14.2*a*, illustrating how the reactance varies as the frequency is varied.

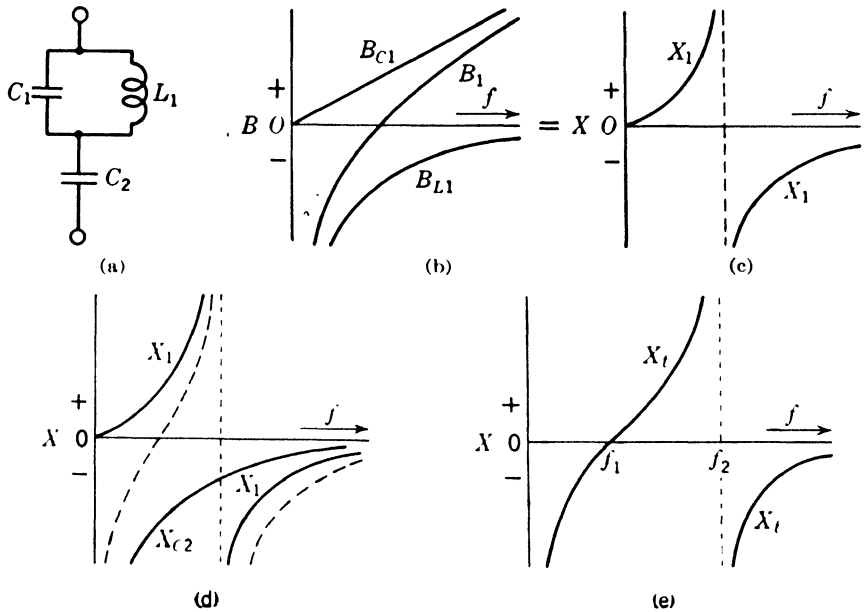
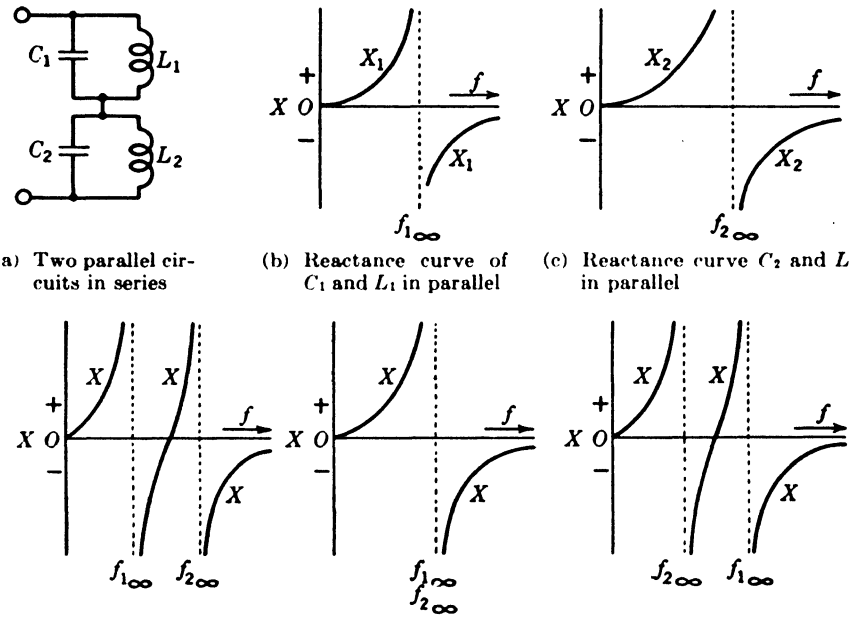


FIG. 14.2. Reactance sketch for a series-parallel circuit.

Analyzing the total reactance curve of Fig. 14.2*e*, it will be noted that the reactance of the series-parallel circuit is negative at low frequencies. In this region, from zero frequency to the series-resonant frequency f_1, the reactance of capacitor C_2 predominates, giving a capacitive reactance. Finally, at f_1, the capacitor C_2 becomes series resonant with inductance L_1 and capacitor C_1 in parallel, and the resulting reactance is zero. From f_1 to f_2, the circuit has inductive reactance varying from zero to infinity (assuming no resistance present in the circuit). At frequency f_2, capacitor C_1 becomes resonant with inductance L_1, giving an antiresonant effect for the series-parallel circuit. As the frequency is increased from f_2 to ∞, the circuit is again capacitive, owing to the capacitor C_1. The reactance decreases from $-\infty$ and approaches zero as the frequency is increased more and more. Summarizing, the effect of capacitor C_2 predominates from zero frequency to f_1; the effect of inductance L_1 predominates from

frequency f_1 to f_2, and finally capacitor C_1 predominates as the frequency is increased above f_2.

The above network of Fig. 14.2*a* is identical with the wave trap of Fig. 14.1*e*. This wave trap will function properly only when the desired frequency f_1 is less than the undesired frequency f_2. An inspection of the total-reactance curve in Fig. 14.2*e* makes this clear. At f_1, series resonance is obtained, hence a high current and a large voltage across C_2 of the desired frequency; at f_2, the drop across the antiresonant circuit is high, and hence the voltage across C_2 is very low.



(a) Two parallel circuits in series

(b) Reactance curve of C_1 and L_1 in parallel

(c) Reactance curve C_2 and L_2 in parallel

(d) Total reactance curve $f_{1\infty} > f_{2\infty}$

(e) Total reactance curve $f_{1\infty} = f_{2\infty}$

(f) Total reactance curve $f_{1\infty} < f_{2\infty}$

FIG. 14.3. Reactance sketch of two parallel circuits connected in series.

14.3. Reactance Sketch of Two Parallel Circuits Connected in Series. The purpose of this section is to point out the value of qualitative sketches and to compare them with the possible results from a quantitative study of the circuit. Adding the susceptance curves of the individual parallel circuits, as shown in Fig. 14.3*a*, and taking their negative reciprocals yields the reactance curves of Figs. 14.3*b* and *c*. Let the antiresonant frequencies of these circuits be $f_{1\infty}$ and $f_{2\infty}$, respectively. When the reactance curves of Figs. 14.3*b* and *c* are added, there are three possible total-reactance curves, as shown in Figs. 14.3*d*, *e*, and *f*. The resulting total-reactance curve will depend upon the quantitative values of C and L used in the two parallel circuits. In general, we can say qualitatively

that the total-reactance curve will have two antiresonant frequencies and a resonant frequency between them. The case with $f_{1_\infty} = f_{2_\infty}$ can be considered as a special case or as an exception to the general rule. Usually the antiresonant frequencies of the various circuits are labeled and carried through the analysis *only* when the circuit is to be solved *quantitatively*.

It is of passing interest to point out that the reactance curves in Fig. 14.3*e* will have a larger slope than the reactance curves of either parallel circuit taken alone. This scheme is sometimes used to increase the selectivity of a circuit.

14.4. Reactance Sketch of a Complex Circuit. Complex circuits are handled in exactly the same way that series, parallel, and series-parallel circuits are handled, the only difference being that more steps must be taken before arriving at the final total-reactance sketch. Figure 14.4 shows the steps involved in solving a more complicated structure. Using the reasoning of the last paragraphs, the student should be able to follow the curves of Fig. 14.4 through the various steps to the final total-reactance sketch for the complete circuit.

Note that the reactance sketch of Fig. 14.4*f* is the same as Fig. 14.2*e*, which applies to the wave trap of Fig. 14.1*e*. This means that a capacitor in parallel with a series combination of a capacitor and an inductance will give the same total attenuation characteristics as a capacitor in series with a parallel combination of an inductance and a capacitor. Of course, the values must be determined so that the desired results will be obtained. The parallel circuit between *c* and *d* of Fig. 14.4*a* would function in a wave trap in such a way that if the antenna and ground and the input terminals to the set were connected across the whole circuit, the series combination of L_1 and C_1 should be tuned to the undesired frequency so as to short it to ground, while the desired frequency would give antiresonance, and hence a high voltage across the combination. In this case, the undesired signal is eliminated by series resonance, and the desired signal obtained from antiresonance—just opposite to the results obtained in Fig. 14.1*e*. However, if the receiver were connected across the series-parallel circuit of Fig. 14.1*e*, then the parallel tuning capacitor would be tuned to the desired signal, and the series capacitor tuned to the undesired signal. Changing the connections in this manner permits the rejection of an undesired frequency either below or above the desired frequency.

14.5. Quantitative Solution of A-c Circuits. The solution of a-c circuits is accomplished in the same way that d-c circuits are solved, with the exception that *vectors* (complex quantities) are used instead of *scalars* (real quantities). This means that the *general form* of Ohm's and

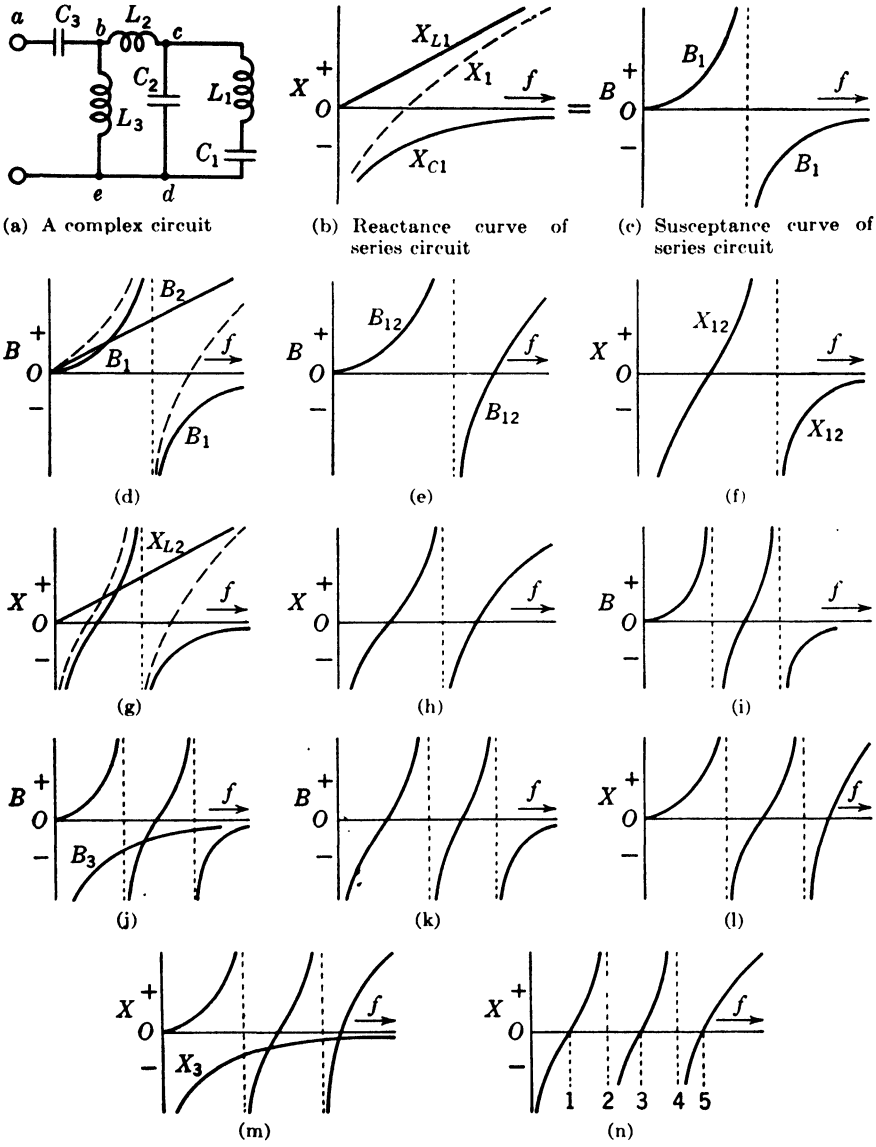


FIG. 14.4. Reactance sketch of a complex circuit.

Kirchhoff's laws must be applied. These important laws can be stated as follows:

Ohm's Law. The vector voltage across an impedance is equal to the product of the vector current and the impedance; thus

$$\dot{E} = \dot{I}\dot{Z} \tag{14.1}$$

In general the impedance Z of Ohm's law will contain various combinations of inductance, capacity, and resistance. This will determine the phase relationship and magnitude of $\dot{E}$ when $\dot{I}$ is given.

Kirchhoff's Voltage Law. The vector voltage across a circuit is equal to the sum of the voltage drops around the circuit; thus

$$\dot{E} = \dot{I}_1\dot{Z}_1 + \dot{I}_2\dot{Z}_2 + \dot{I}_2\dot{Z}_3 + \cdots \tag{14.2}$$

Two alternate ways of stating this law are (1) the sum of the vector voltages around a circuit is zero, or (2) the sum of the vector voltages around a circuit forms a closed polygon.

Kirchhoff's Current Law. The vector current flowing up to a junction equals the sum of the vector currents flowing away from the junction; thus

$$\dot{I} = \dot{I}_1 + \dot{I}_2 + \dot{I}_3 + \cdots \tag{14.3}$$

Two alternate ways of stating this law are (1) the sum of the vector currents flowing up to a junction is zero, or (2) the sum of the vector currents flowing up to a junction forms a closed polygon.

Ohm's law of Eq. (14.1) was utilized in Eq. (14.2) to express the voltage drops across the various elements. The voltage across $\dot{Z}_1$ is $\dot{I}_1\dot{Z}_1$ where $\dot{I}_1$ is the current flowing through Z_1. Similarly, the voltage across $\dot{Z}_2$ is $\dot{I}_2\dot{Z}_2$ where $\dot{I}_2$ is the current flowing through $\dot{Z}_2$. If there is no current junction between $\dot{Z}_1$ and $\dot{Z}_2$, then $\dot{I}_1 = \dot{I}_2$, and the impedances $\dot{Z}_1$ and $\dot{Z}_2$ can be added vectorially to simplify the equation. In a like manner, the current through the other impedances will be the same if there are no current junctions between them. For simplicity, only three impedance elements were given. The $(+ \cdots)$ at the end of the equation means that more voltages can be added if there are more than three impedances in the circuit.

The voltage $\dot{E}$ on the left-hand side of Eq. (14.2) is a voltage rise, while the voltages on the right-hand side are voltage drops. If there is more than one voltage rise in the circuit, they can be added to $\dot{E}$ on the left-hand side of the equation. This equation can be simplified by expressing all the voltage rises as negative voltage drops and simply adding them to the actual voltage drops. Then the sum of the voltage drops, which includes the negative voltage rises, is zero. This is accomplished mathematically by simply shifting all the terms of Eq. (14.2) to the left-hand side of the equation and multiplying through by -1.

Kirchhoff's current law expressed in Eq. (14.3) shows only one current $\dot{I}$ flowing up to the junction and any number of currents flowing away from the junction. If there were more currents flowing up to the junction, they should be added to $\dot{I}$ on the left-hand side of the equation. This equation can be simplified by placing all the currents on the left-

hand side; then their sum is zero. The current $\dot{I}_1$, for example, flowing away from the point is then expressed as $-I_1$ flowing up to the point.

A complete understanding of these important a-c laws can best be achieved by solving a variety of problems in which they are applicable.

14.6. Solution of Series-parallel Circuits Containing Pure Reactance Elements. The quantitative solution of a-c circuits is materially simplified if the circuit contains pure reactance elements, just as the analysis is simplified for pure resistive elements. The reason for this is that the reactance vectors lie along the y-axis just as the resistance vectors lie along the x-axis. This eliminates the work incidental to changing complex quantities from rectangular to polar form, and vice versa. All the reactance quantities can be expressed in rectangular form and left in that form throughout the solution.

Example 14.1. Referring to Fig. 14.2*a*, determine the equivalent impedance $\dot{Z}_{eq}$ when

$$L_1 = 60\ \mu\text{h}, \qquad C_2 = 200\ \mu\mu\text{f}$$
$$C_1 = 50\ \mu\mu\text{f}, \qquad f = 796\ \text{kc}$$

Solution

$$\omega = 2\pi f = (6.2832)(796{,}000) = 5 \times 10^6$$
$$Z_{L1} = j2\pi f L_1 = j(5 \times 10^6)(60 \times 10^{-6}) = j300 \text{ ohms}$$
$$Z_{C1} = \frac{1}{j2\pi f C_1} = -j\frac{1}{(5 \times 10^6)(50 \times 10^{-12})}$$
$$= -j\frac{10^6}{250} = -j4{,}000 \text{ ohms}$$
$$Z_{C2} = \frac{1}{j2\pi f C_2} = -j\frac{1}{(5 \times 10^6)(200 \times 10^{-12})}$$
$$= -j\frac{10^6}{1{,}000} = -j1{,}000$$

The equivalent impedance of the combination is the sum of the series impedance and the impedance of the parallel combination; thus

$$Z_{eq} = Z_{C2} + \frac{Z_{C1}Z_{L1}}{Z_{L1} + Z_{C1}} \tag{14.4}$$

Substituting

$$Z_{eq} = -j1{,}000 + \frac{(-j4{,}000)(j300)}{j300 - j4{,}000} = -j1{,}000 + \frac{1{,}200{,}000}{-j3{,}700}$$
$$= -j1{,}000 + j324.324 = -j675.676 \text{ ohms} = \text{capacitive reactance} \quad \textit{Ans.}$$

The total-reactance curve of Fig. 14.2*e* shows that 796 kc is between zero frequency and f_1, because the reactance is capacitive and the effect of C_2 predominates.

Example 14.2. Determine f_1 and f_2 as shown in Fig. 14.2*e* for Example 14.1.

Solution. At frequency f_1, the equivalent impedance of the series-parallel circuit is zero; hence Eq. (14.4) can be written

$$Z_{eq} = Z_{C2} + \frac{Z_{C1}Z_{L1}}{Z_{L1} + Z_{C1}} = 0 \tag{14.5}$$

This equation can be written

$$Z_{L1}Z_{C2} + Z_{C1}Z_{C2} + Z_{C1}Z_{L1} = 0 \tag{14.6}$$

In this equation f_1 is the only unknown. The various impedances can be written

$$Z_{L1} = j2\pi f_1 L_1 = j(6.28)(60 \times 10^{-6})f_1 = jf_1(377 \times 10^{-6})$$

$$Z_{C1} = \frac{1}{j2\pi f_1 C_1} = -j\frac{10^{12}}{f_1(6.28)(50)} = \frac{-j10^{12}}{f_1(314.159)}$$

$$Z_{C2} = \frac{1}{j2\pi f_1 C_2} = -j\frac{10^{12}}{f_1(6.28)(200)} = \frac{-j10^{12}}{f_1(1,256.64)}$$

Then the desired products become

$$Z_{L1}Z_{C2} = [jf_1(377 \times 10^{-6})]\left[\frac{-j10^{12}}{f_1(1,256.64)}\right] = 0.3 \times 10^6$$

$$Z_{C1}Z_{C2} = \left[\frac{-j10^{12}}{f_1(314.159)}\right]\left[\frac{-j10^{12}}{f_1(1,256.64)}\right] = -\frac{2.532 \times 10^{18}}{f_1^2}$$

$$Z_{L1}Z_{C1} = [jf_1(377 \times 10^{-6})]\left[\frac{-j10^{12}}{f_1(314.159)}\right] = 1.2 \times 10^6$$

Eq. (14.6) can now be written

$$(0.3 \times 10^6) - \frac{2.532 \times 10^{18}}{f_1^2} + (1.2 \times 10^6) = 0$$

Solving

$$f_1^2 = \frac{2.532}{1.5} \times 10^{12} = 1.688 \times 10^{12}$$

$$f_1 = 1.298 \times 10^6 \qquad \text{(for series resonance)} \qquad \textit{Ans.}$$

Checking

$$\omega_1 = 2\pi f_1 = (6.28)(1.298 \times 10^6) = 8.16 \times 10^6$$

$$Z_{L1} = +j\omega L_1 = +j(8.16)(60) = +j489.6 \text{ ohms}$$

$$Z_{C1} = \frac{1}{+j\omega C_1} = -j\frac{10^6}{(8.16)(50)} = -j2{,}450 \text{ ohms}$$

$$Z_{C2} = \frac{1}{+j\omega C_2} = -j\frac{10^6}{(8.16)(200)} = -j613 \text{ ohms}$$

$$\frac{Z_{L1}Z_{C1}}{Z_{L1} + Z_{C1}} = \frac{(+j489.6)(-j2{,}450)}{+j489.6 - j2{,}450} = \frac{1{,}200{,}000}{-j1{,}960} = +j613 \text{ ohms}$$

Hence

$$Z_{C2} + \frac{Z_{L1}Z_{C1}}{Z_{L1} + Z_{C1}} = -j613 + j613 = 0 \qquad \textit{Check}$$

Solving for frequency f_2, the parallel circuit must be antiresonant, and for this condition

$$Z_{L1} + Z_{C1} = 0 \tag{14.7}$$

As before

$$Z_{L1} = j2\pi f_2 L_1 = jf_2(377 \times 10^{-6})$$

and

$$Z_{C1} = \frac{1}{j2\pi f_2 C_1} = \frac{-j10^{12}}{f_2(314.159)}$$

Substituting in Eq. (14.7),

$$+jf_2(377 \times 10^{-6}) - \frac{j10^{12}}{f_2(314.159)} = 0$$

Transposing and multiplying through by $314.159f_2$ gives

$$f_2^2 = \frac{3{,}184 \times 10^6}{377 \times 10^{-6}} = 8.45 \times 10^{12}$$

Extracting the square root,

$$f_2 = 2{,}908 \text{ kc} \qquad \text{(for antiresonance)} \qquad \textit{Ans.}$$

Checking,

$$Z_{L1} = +j(6.28)(2.908 \times 10^6)(60 \times 10^{-6}) = +j1{,}096 \text{ ohms}$$

$$Z_{C1} = \frac{1}{+j(6.28)(2.908 \times 10^6)(50 \times 10^{-12})} = \frac{10^6}{+j913} = -j1{,}096$$

Hence $$Z_{L1} + Z_{C1} = +j1{,}096 - j1{,}096 = 0 \qquad \textit{Check}$$

It will be noted that in the preceding solutions polar coordinates were not used. When dealing with pure reactance elements, there is no advantage in using the polar form because all quantities would be $+j = +90°$ or $-j = -90°$ if the notation were changed from rectangular form to polar form.

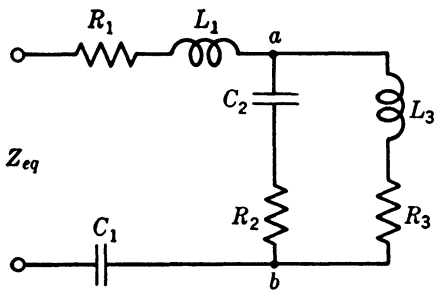


FIG. 14.5. A series-parallel circuit containing both resistance and reactance.

14.7. Solution of Series-parallel Circuits Containing Resistance and Reactance. The quantitative solution of series-parallel circuits containing both resistance and reactance necessitates the use of the general form of Ohm's and Kirchhoff's laws. This means that the impedances are complex quantities and are conveniently added or subtracted in rectangular form, while for multiplication or division the polar form is more suitable.

Example 14.3. In Fig. 14.5 let

$f = 796$ kc, $C_1 = 3{,}155$ $\mu\mu$f
$R_1 = 50$ ohms, $C_2 = 3{,}333$ $\mu\mu$f
$R_2 = 80$ ohms, $L_1 = 30$ μh
$R_3 = 50$ ohms, $L_3 = 17.32$ μh

Determine the equivalent impedance.

Solution. Solving for the angular velocity

$$\omega = 2\pi f = (6.28)(796 \times 10^3) = 5 \times 10^6$$

The values of the reactance elements are

$$X_{c1} = \frac{1}{\omega C_1} = \frac{1}{(5 \times 10^6)(3{,}155 \times 10^{-12})} = 63.4 \text{ ohms}$$

$$X_{c2} = \frac{1}{\omega C_2} = \frac{1}{(5 \times 10^6)(3{,}333 \times 10^{-12})} = 60 \text{ ohms}$$

$$X_{L1} = \omega L_1 = (5 \times 10^6)(30 \times 10^{-6}) = 150 \text{ ohms}$$

$$X_{L2} = \omega L_2 = (5 \times 10^6)(17.32 \times 10^{-6}) = 86.6 \text{ ohms}$$

The impedance elements can now be written in rectangular and polar form as follows:

$$Z_1 = R_1 + j(X_{L1} - X_{C1}) = 50 + j86.6 = \underline{100/60°}$$
$$Z_2 = R_2 - jX_{C2} = 80 - j60 = \underline{100/-36°52'}$$
$$Z_3 = R_3 + jX_{L3} = 50 + j86.6 = \underline{100/60°}$$

The impedance of the parallel circuit is

$$Z_p = \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{(100/-36°52')(100/+60°)}{80 - j60 + 50 + j86.6}$$
$$= \frac{10{,}000/27°08'}{130 + j26.6} = \frac{10{,}000/27°08'}{152.7/11°34'}$$
$$= 75.4/15°34' = 75.4(0.9633 + j0.2684)$$
$$= 72.6 + j20.33$$

The equivalent impedance of the series-parallel circuit can be written

$$Z_{eq} = Z_1 + Z_p = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} \tag{14.8}$$

Substituting in this equation yields

$$Z_{eq} = 50 + j86.6 + 72.6 + j20.23$$
$$= 122.6 + j106.83 = \underline{162.7/41°06'} \qquad Ans.$$

This equivalent impedance made up of a resistance and reactance component could be connected in series to replace the series-parallel circuit of Fig. 14.5 at 796 kc. If the frequency is changed, the equivalent impedance also changes.

Equation (14.8) is a general equation that can be used in any series-parallel circuit. More complicated series-parallel networks are solved by the same procedure but involve more steps.

Some networks cannot be easily simplified by the methods given. In such cases equivalent T- and π-networks are of great value. This will be treated in the next chapter.

Summary

Combinations of series-resonant and parallel-resonant circuits can be used to obtain frequency-discrimination performance needed in wave traps, filters, and equalizers.

In complex series-parallel circuits, the necessarily positive slope of the reactance curve requires that resonant and antiresonant frequencies alternate. Thus, a resonant frequency must always occur between two antiresonant frequencies and an antiresonant frequency between two resonant frequencies.

The reactance curve of two parallel-resonant circuits in series and tuned to the same frequency will have a steeper slope than the curve of only one circuit. The same is true for two series-resonant circuits in parallel. This is a very useful characteristic in the design of filters.

Exercises

14.1. What is a wave trap? Explain its action.

14.2. Draw three wave-trap circuits and their reactance sketches.

14.3. Draw the reactance sketch of the circuit in Fig. 14.6. How many resonant points may be present? How many antiresonant points?

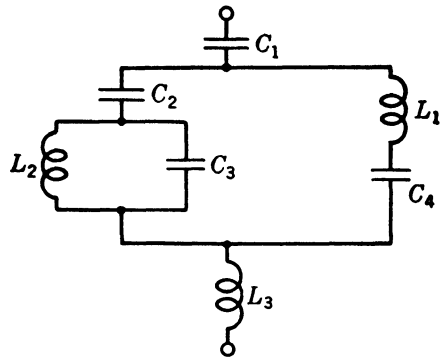


FIG. 14.6. Series-parallel circuit.

14.4. Considering the reactance elements of Fig. 14.6 to be pure and of the following values, determine the equivalent reactance of the network:

$X_{C1} = 50$ ohms, $X_{L1} = 100$ ohms
$X_{C2} = 40$ ohms, $X_{L2} = 20$ ohms
$X_{C3} = 60$ ohms, $X_{L3} = 50$ ohms
$X_{C4} = 70$ ohms

14.5. Determine the equivalent impedance of Fig. 14.5 when

$f = 1{,}592$ kc, $C_1 = 3{,}155$ $\mu\mu$f
$R_1 = 50$ ohms, $C_2 = 3{,}333$ $\mu\mu$f
$R_2 = 80$ ohms, $L_1 = 30$ μh
$R_3 = 50$ ohms, $L_3 = 17.32$ μh

CHAPTER 15

NETWORK THEOREMS

In dealing with complicated electrical networks, the fundamental problem that is encountered over and over again is that of finding the current that will flow in one branch of the circuit when a voltage is impressed in another branch. There are many ways of approaching this problem, some of which have already been considered. The solution in the general case, while not difficult, is sometimes extremely complicated. For this reason a number of theorems have been developed, the application of which can materially reduce the amount of labor involved and often enable a solution to be reached in a relatively short time.

Over and above this practical application of these theorems, they are often useful in theoretical work by enabling a general network to be handled easily, no matter what its complexity.

There is an absolutely general method of solving all network problems through the use of determinants, but it is of a somewhat advanced nature and will not be considered here. For the same reason, complete and general proofs of the theorems will not be given, but examples will be provided that should give some idea to the student of how the proof in the general case would proceed.

15.1. The Superposition Theorem. Implicit in all the work that has been done with networks up to this point is the *superposition theorem*, which states, in essence, that causes and effects in a network composed of linear elements are independent and additive.

Superposition Theorem. If a voltage E_1 in series with branch A_1 in a network causes a current I_1 to flow through branch B_1 of the network, and if a voltage E_2 in branch A_2 (which may be identical with A_1) of the network causes a current I_2 to flow through branch B_1 of the network, then if voltage E_1 in branch A_1 and voltage E_2 in branch A_2 operates simultaneously, the current that will flow through branch B_1 will equal $I_1 + I_2$.

The condition necessary in order that the superposition theorem should apply is that all the elements in the network, in general, should be linear, or "ohmic." In other words, each element in the network must obey Ohm's law. An *ohmic*, or *linear, element* is one whose resistance or impedance is independent of the current through it, or the voltage across it. If the elements of a network are not linear, the superposition theorem will not, in general, hold true.

The usefulness of the superposition theorem lies in the fact that if a network is given that contains a number of voltage sources, the current that will flow in any branch of the network as a result of these various emfs can be found by solving for the current in this branch due to each of the separate emfs, and then adding all the currents thus found to find the resultant current. In carrying through a solution by means of the superposition theorem, all generators but the one under consideration are short-circuited, if of zero internal impedance, or are replaced by their internal impedance if this differs from zero.

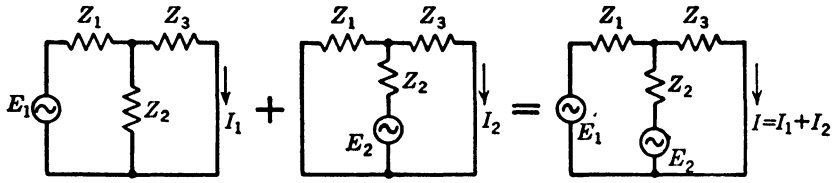


FIG. 15.1. Circuits illustrating the superposition theorem.

Example 15.1. Find the current I that will flow through Z_3 in Fig. 15.1, under the simultaneous influence of E_1 and E_2. Both generators are of zero internal impedance.

Solution. I can be found by first solving for the current I_1 that will flow through Z_3 when E_2 is short-circuited, then solving for the current I_2 that will flow through Z_3 when E_1 is short-circuited, and adding I_1 and I_2 to find the total current that will flow through Z_3 when both generators are operating.

The current that will flow through Z_3 owing to E_1 when E_2 is short-circuited is given by

$$I_1 = \frac{E_1 Z_2}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \tag{15.1}$$

The current I_2, due to E_2, that will flow through Z_3 when E_1 is short-circuited is given by

$$I_2 = \frac{E_2 Z_1}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \tag{15.2}$$

Therefore, the total current I that will flow through Z_3 when E_1 and E_2 are impressed on the network simultaneously as indicated will be given by the sum of I_1 and I_2, or

$$I = \frac{E_1 Z_2 + E_2 Z_1}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} \qquad \textit{Ans.} \tag{15.3}$$

The expressions for I_1 and I_2 given in Eqs. (15.1) and (15.2) can be found by the methods of the previous chapter.

As long as the elements of a network obey Ohm's law, the superposition theorem applies in an absolutely general manner. Not only may the voltage sources be of different magnitudes and placed at different points in the network, but they may be of different frequency and phase, and the superposition theorem will still apply.

15.2. Equivalent Networks. A concept of great utility in the solution and investigation of networks is the idea of *equivalence*. This concept has been used many times in the preceding material in this book but has not been dealt with explicitly. Two networks are said to be *equivalent* with respect to one or more sets of terminals if either can be substituted for the other without affecting the voltages, currents, and impedances that they present to these terminals. Equivalent networks have been dealt with extensively in previous chapters where the equivalence of T- and π-sections under certain conditions has been demonstrated. A T-section, for example, will be equivalent to a π-section as far as its input and output terminals are concerned, if it can be substituted for the π-section in a network without causing any change in the voltages and currents at these terminals.

This is not to say that a T-section is the only possible equivalent of a given π-section. In general, there will be an infinite number of sections equivalent to a given section with respect to any pair, or number of pairs, of terminals. That this is true can be seen very quickly by considering that any single impedance in a network can always be replaced by two other impedances in parallel or in series, whose total impedance equals the original impedance of the element. Thus, there is no limit to the number of networks that can be derived from a given network and still be exactly equivalent as far as behavior with respect to any given pair of terminals is concerned.

15.3. Equivalent Voltage and Current Sources. Since actual physical sources of electrical energy have internal impedance, the terminal voltage is a function of the terminal current, and conversely, the terminal current is a function of the terminal voltage. In some cases it is advantageous to treat a physical source as a voltage source, while in other cases it is better to convert it to an equivalent current source. In general, the voltage source is best suited to the application of Kirchhoff's voltage law for equating the voltage drops around a closed circuit, while the current source simplifies solutions involving the application of Kirchhoff's current law to equate the currents at a junction.

A voltage source can be represented, as shown in Fig. 15.2*a*, by a constant-voltage generator E in series with an internal impedance Z. By the application of Kirchhoff's voltage law, the terminal voltage is given by

$$V_t = E - ZI_t \tag{15.4}$$

where V_t = terminal voltage of source (function of I_t)
I_t = terminal current of source (function of V_t)
E = internal voltage of source (independent of V_t and I_t)
Z = internal impedance of voltage source

A current source can be represented, as shown in Fig. 15.2*b*, by a constant-current generator I in parallel with an internal admittance Y. By the application of Kirchhoff's current law, the terminal current is

$$I_t = I - YV_t \qquad (15.5)$$

where I = internal current of source (independent of I_t and V_t)
Y = internal admittance of current source and the other values are as defined in Eq. (15.4).

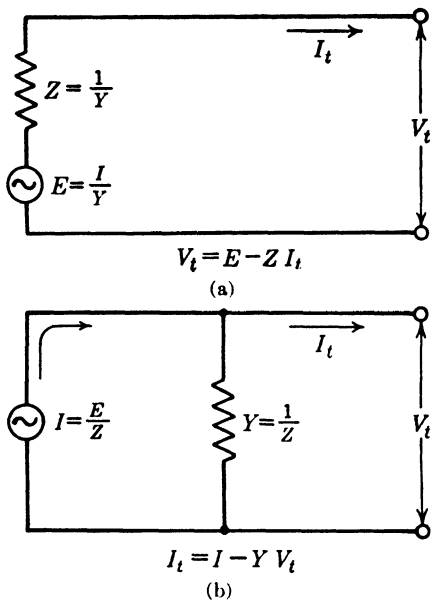


FIG. 15.2. Equivalent voltage and current sources.

The equivalence between these two circuits can be shown by the following conversion relations:

$$E = IZ \qquad (15.6)$$
$$I = EY \qquad (15.7)$$
$$ZY = 1 \qquad (15.8)$$

Solving Eq. (15.4) for I_t and substituting from the conversion relations results in Eq. (15.5) as follows:

$$I_t = \frac{E}{Z} - \frac{V_t}{Z} = I - YV_t \qquad (15.9)$$

Similarly, solving Eq. (15.5) for V_t and substituting from the conversion relations results in Eq. (15.4).

$$V_t = \frac{I}{Y} - \frac{I_t}{Y} = E - ZI_t \qquad (15.10)$$

It is important to note that the efficiencies, and hence power losses in the generators, of equivalent voltage and current sources are the same only when the load impedance is matched to the generator impedance to give 50 per cent efficiency. A voltage source is ideal for open-circuit operation because the efficiency is 100 per cent. The current source is not suited for open-circuit operation because the efficiency is 0 per cent. Conversely, a current source is ideal for short-circuit operation because the efficiency is 100 per cent. The voltage source is not suited for short-circuit operation because the efficiency is 0 per cent. In general, voltage sources are more efficient when connected to high-impedance loads, and

current sources are more efficient when connected to low-impedance loads. It may be dangerous to short-circuit an efficient voltage source because of the low internal impedance and the resulting high terminal current. Likewise, it may be dangerous to open-circuit an efficient current source because of the low internal admittance and the resulting high terminal voltage. The internal impedance of E in the voltage source is zero, while the internal admittance of I in the current source is zero.

Example 15.2. Show that the network of Fig. 15.3*b* is equivalent to the network of Fig. 15.3*a* for the constants given. Determine the currents I_1 and I_2 in Fig. 15.3*a* and the voltages V_{12} and V_{43} in Fig. 15.3*b* by the application of Kirchhoff's voltage and current laws.

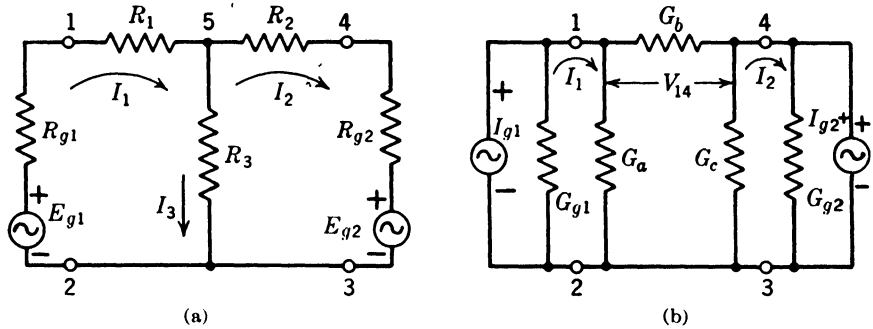


FIG. 15.3. Equivalent voltage and current sources connected to equivalent T- and π-sections.

Solution. By the application of Eq. (15.8), for this simple case of pure resistances instead of complex impedances, and pure conductances instead of complex admittances, the equivalent generator admittances are

$$G_{g1} = \frac{1}{R_{g1}} = \frac{1}{0.1} = 10 \text{ mhos}$$
$$G_{g2} = \frac{1}{R_{g2}} = \frac{1}{0.2} = 5 \text{ mhos}$$

Similarly, by Eq. (15.7) the equivalent generator currents are

$$I_{g1} = E_{g1}G_{g1} = (2)(10) = 20 \text{ amp}$$
$$I_{g2} = E_{g2}G_{g2} = (4)(5) = 20 \text{ amp}$$

The equivalent π-section conductances can be secured by first converting the T-section to a π-section and then taking the reciprocal of the resistances thus

$$G_a = \frac{1}{R_a} = \frac{R_2}{R_1R_2 + R_2R_3 + R_3R_1} = \frac{20}{(10)(20) + (20)(5) + (5)(10)}$$
$$= \frac{20}{350} = 0.0571 \text{ mho}$$
$$G_b = \frac{1}{R_b} = \frac{R_3}{R_1R_2 + R_2R_3 + R_3R_1} = \frac{5}{350} = 0.0143 \text{ mho}$$
$$G_c = \frac{1}{R_c} = \frac{R_1}{R_1R_2 + R_2R_3 + R_3R_1} = \frac{10}{350} = 0.0286 \text{ mho}$$

In Fig. 15.3*a*, the joint or branch node currents at junction or node 5 by Kirchhoff's current law are

$$I_1 = I_2 + I_3$$

The loop voltages around the two meshes by Kirchhoff's voltage law are

$$I_1R_{g1} + I_1R_1 + I_3R_3 = E_{g1}$$
$$I_2R_{g2} + I_2R_2 - I_3R_3 = -E_{g2}$$

Substituting for I_3 from the equation of branch currents at node 5, these loop-voltage equations can be written

$$I_1(R_{g1} + R_1 + R_3) - I_2R_3 = E_{g1}$$
$$-I_1R_3 + I_2(R_{g2} + R_2 + R_3) = -E_{g2}$$

Substituting for the values of resistance and voltage in these loop voltage equations gives

$$15.1I_1 - 5I_2 = 2$$
$$-5I_1 + 25.2I_2 = -4$$

The solution of these simultaneous equations by determinants results in

$$I_1 = \frac{\begin{vmatrix} 2 & -5 \\ -4 & 25.2 \end{vmatrix}}{\begin{vmatrix} 15.1 & -5 \\ -5 & 25.2 \end{vmatrix}} = \frac{50.4 - 20}{380.5 - 25}$$
$$= \frac{30.4}{355.5} = 0.0855 \text{ amp} \qquad \textit{Ans.}$$

$$I_2 = \frac{\begin{vmatrix} 15.1 & 2 \\ -5 & -4 \end{vmatrix}}{\begin{vmatrix} 15.1 & -5 \\ -5 & 25.2 \end{vmatrix}} = \frac{-60.4 + 10}{380.5 - 25}$$
$$= \frac{-50.4}{355.5} = -0.0142 \text{ amp} \qquad \textit{Ans.}$$

Now, in the network of Fig. 15.3*b*, the loop voltage around the terminals, by Kirchhoff's voltage law, can be written

$$V_{12} = V_{14} + V_{43}$$

The branch currents at nodes 1 and 4, by Kirchhoff's current law, are

$$V_{12}G_{g1} + V_{12}G_a + V_{14}G_b = I_{g1}$$
$$V_{43}G_{g2} + V_{43}G_c - V_{14}G_b = I_{g2}$$

Substituting for V_{14} from the loop-voltage equation, these node-current equations can be written

$$V_{12}(G_{g1} + G_a + G_b) - V_{43}G_b = I_{g1}$$
$$-V_{12}G_b + V_{43}(G_{g2} + G_c + G_b) = I_{g2}$$

Substituting for the values of conductance and current in these node-current equations gives

$$10.0714V_{12} - 0.0143V_{43} = 20$$
$$-0.0143V_{12} + 5.0429V_{43} = 20$$

The solution of these equations by determinants results in

$$V_{12} = \frac{\begin{vmatrix} 20 & -0.0143 \\ 20 & 5.0429 \end{vmatrix}}{\begin{vmatrix} 10.0714 & -0.0143 \\ -0.0143 & 5.0429 \end{vmatrix}} = \frac{100.858 + 0.286}{50.789 - 0.0002}$$

$$= \frac{101.144}{50.789} = 1.99 \text{ volts} \qquad \textit{Ans.}$$

$$V_{43} = \frac{\begin{vmatrix} 10.0714 & 20 \\ -0.0143 & 20 \end{vmatrix}}{\begin{vmatrix} 10.0714 & -0.0143 \\ -0.0143 & 5.0429 \end{vmatrix}} = \frac{201.428 + 0.286}{50.789 - 0.0002}$$

$$= \frac{201.714}{50.789} = 3.97 \text{ volts} \qquad \textit{Ans.}$$

It is worth noting that the solution of the π-section with current sources is as easy as the solution of the T-section with voltage sources. Much similarity will be observed in the two solutions. The node-current equations of one network resemble the loop-voltage equations of the other network, and vice versa.

15.4. Thévenin's Theorem. Another network theorem of wide general usefulness is *Thévenin's theorem.* By means of this theorem, any network, no matter how complicated, containing any number of generators, can, as far as its output terminals are concerned, be replaced by a single generator in series with a single impedance. It is not even necessary that the elements of the circuit be linear. In this respect, at least, Thévenin's theorem is even more general than the superposition theorem.

Thévenin's Theorem. With respect to a given pair of terminals, any network containing any number of generators can be replaced by a single generator in series with a single impedance connected across the given terminals. The voltage of the generator is given by the open-circuit voltage that would be measured across the terminals in the absence of a load, and the impedance is given by the impedance that is seen looking back into the terminals when all generators are replaced by their internal impedances.

Example 15.3. What current I will flow through Z_5 in the unbalanced bridge circuit of Fig. 15.4?

Solution. The solution of this problem by ordinary methods is rather tedious, but by Thévenin's theorem it becomes extremely simple. According to this theorem, the network, *as far as the terminals a and b are concerned,* can be replaced by a single generator E' in series with a single impedance Z' connected across a and b, as shown in Fig. 15.4*b*.

To find the voltage E' of the equivalent circuit, the network is redrawn as in Fig. 15.4*c*. The voltage E' will be given by the voltage that would exist between the points a and b if Z_5 were removed. This voltage can be found in a number of ways, but possibly the easiest way is to consider that the point c is grounded and then find E' as the difference between the voltages at the points a and b above ground.

Z_1 and Z_2, and Z_3 and Z_4 constitute two voltage dividers connected across E, and the voltages E_1 and E_2 at points a and b are consequently given quite simply by

$$E_1 = \frac{EZ_2}{Z_1 + Z_2} \tag{15.11}$$

$$E_2 = \frac{EZ_4}{Z_3 + Z_4} \tag{15.12}$$

E' will be the difference between these two voltages. It makes no difference which voltage is subtracted from the other. If a choice is made that results in a negative current in the answer, it will simply mean that the current that will flow through Z_5 will be opposite to the assumed direction. We will assume that E_1 is larger than E_2, or in other words, that the current through Z_5 will flow from a to b. E' will therefore be given by

$$\begin{aligned} E' &= E_1 - E_2 \\ &= E\left(\frac{Z_2}{Z_1 + Z_2} - \frac{Z_4}{Z_3 + Z_4}\right) \end{aligned} \tag{15.13}$$

To find the impedance Z' in Fig. 15.4*b*, the generator E in Fig. 15.4*c* [or 15.4*a*] is replaced by a short circuit, and the impedance looking into terminals a and b is computed [Fig. 15.4*d*]. This figure can be redrawn as in Fig. 15.4*e*, as can be seen very easily. From Fig. 15.4*e*, the impedance that will be seen looking back into terminals a and b is given immediately as Z_1 in parallel with Z_2, plus Z_3 in parallel with Z_4, or

$$Z' = \frac{Z_1Z_2}{Z_1 + Z_2} + \frac{Z_3Z_4}{Z_3 + Z_4} \tag{15.14}$$

The current I that will flow through impedance Z_5 in Fig. 15.4*b* is given by

$$I = \frac{E'}{Z' + Z_5} \tag{15.15}$$

All that is necessary is to substitute the values of E' and Z', which have just been found, in Eq. (15.15), and the solution is immediate.

$$\begin{aligned} I &= \frac{E\left(\dfrac{Z_2}{Z_1 + Z_2} - \dfrac{Z_4}{Z_3 + Z_4}\right)}{\dfrac{Z_1Z_2}{Z_1 + Z_2} + \dfrac{Z_3Z_4}{Z_3 + Z_4} + Z_5} \\ &= \frac{E[Z_2(Z_3 + Z_4) - Z_4(Z_1 + Z_2)]}{Z_1Z_2(Z_3 + Z_4) + Z_3Z_4(Z_1 + Z_2) + Z_5(Z_1 + Z_2)(Z_3 + Z_4)} \\ &= \frac{E(Z_2Z_3 - Z_1Z_4)}{Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4 + Z_1Z_3Z_5 + Z_1Z_4Z_5 + Z_2Z_3Z_5 + Z_2Z_4Z_5} \end{aligned}$$

Ans. (15.16)

In the event that the network contains a number of generators, the voltage E' will be given as the sum of the various open-circuit voltages across the output terminals, each of which is found by considering each of the generators of the network separately. The impedance Z' will, of course, be the same no matter how many generators are in the network. Another form of Thévenin's theorem, is as follows:

15.5. Norton's Theorem. Another theorem, which is in reality another form of Thévenin's theorem, is as follows:

Norton's Theorem. With respect to any pair of output terminals, any network containing any number of generators and impedances can be replaced by a single constant-current generator in parallel with a single impedance connected across the given output terminals. The current produced by the generator is given by the current that would flow between the output terminals if they were short-circuited, and the impedance is given by the impedance that is seen looking back into the output terminals when all generators in the circuit are replaced by their internal impedances.

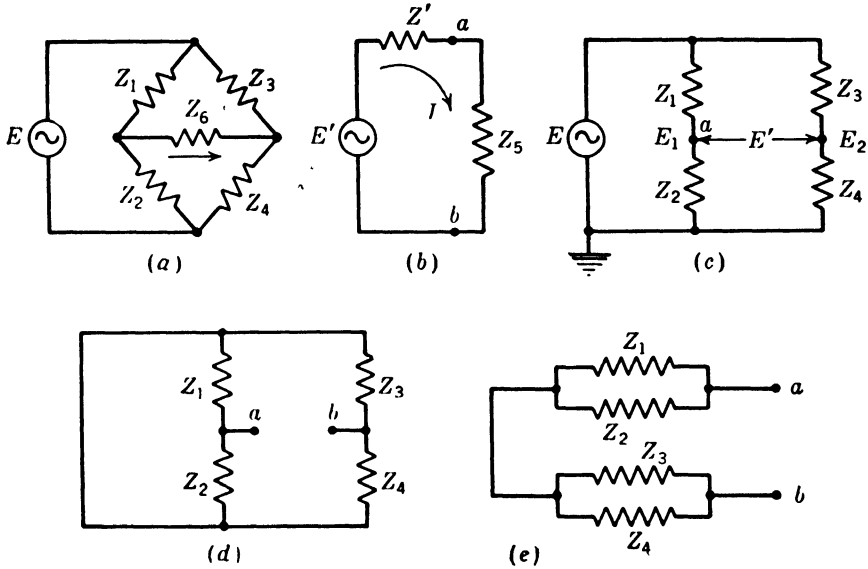


FIG. 15.4. Circuits to illustrate Thévenin's theorem.

A *constant-current* generator is a generator that will deliver a constant current irrespective of the load that is connected across its terminals, and it should be carefully distinguished from a *constant-voltage* generator, which produces a constant voltage at its terminals irrespective of the load that is connected across them. In reality, of course, no generator is ever actually a constant-current generator, any more than any generator is ever a constant-voltage generator, but the concepts of constant-current and constant-voltage generators are extremely useful in many applications. Certain vacuum tubes such as pentodes and photocells, for instance, behave like constant-current generators over very wide ranges, and in dealing with these tubes, under most conditions it is permissible to consider them as generators that will deliver a certain current to a load, independent of the magnitude of the load.

Example 15.4. Solve the network of Example 15.2 by means of Norton's theorem.
Solution. According to Norton's theorem, the network of Fig. 15.4*a* can be

replaced, *so far as terminals a and b are concerned,* by the network of Fig. 15.5. The impedance Z' in this figure is the same as that which was found in Example 15.2 and is given by

$$Z' = \frac{Z_1 Z_2}{Z_1 + Z_2} + \frac{Z_3 Z_4}{Z_3 + Z_4} \tag{15.17}$$

The current of the constant-current generator, I', will be given by the current that would flow between the terminals a and b when these are short-circuited. This can be found very easily from Eq. (15.15) by setting Z_5 equal to zero, or in other words

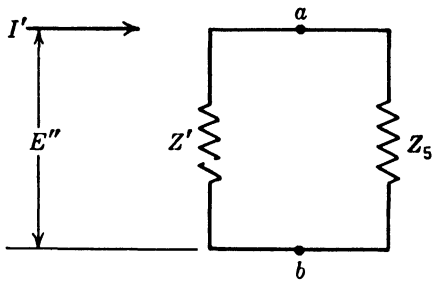


FIG. 15.5. Circuit to illustrate Norton's theorem.

$$I' = \frac{E'}{Z'} \tag{15.18}$$

Referring to Fig. 15.5, the current I flowing through Z' and Z_5 in parallel will generate a voltage E'' across the combination, which is given by

$$E'' = I' \frac{Z' Z_5}{Z' + Z_5} \tag{15.19}$$

Then the current I_{Z5} that will flow through Z_5 under the influence of the voltage E'' will be given by

$$I_{Z5} = I' \frac{Z'}{Z' + Z_5} \tag{19.20}$$

Since I' is given by Eq. (15.18) as

$$I' = \frac{E'}{Z'}$$

Eq. (15.20) can be rewritten as

$$I_{Z5} = \left(\frac{E'}{Z'}\right) \frac{Z'}{Z' + Z_5}$$

$$\therefore = \frac{E'}{Z' + Z_5} \tag{15.21}$$

which is seen to be identical with Eq. (15.15) of the preceding example.

15.6. Reciprocity Theorem. Another theorem of considerable interest and utility is the *reciprocity theorem,* which is as follows:

Reciprocity Theorem. If a voltage E in a branch A of a network causes a current I to flow in another branch B of the network, then if the same voltage E is introduced in the branch B of the network, the same current I will flow in branch A.

The validity of the reciprocity theorem derives from certain symmetrical relationships that hold in any network composed of linear elements, that is, in any network for which the superposition theorem holds. This theorem can be proved very easily by taking advantage of the fact that any network can be reduced to an equivalent T-section *with respect to any two pairs of terminals.*

Referring to Fig. 15.6*a*, the network in question can be considered to have been replaced by the T-section $Z_1Z_2Z_3$, as far as the terminal pairs 1 and 2 and 3 and 4 are concerned. The current I_1 that will flow through

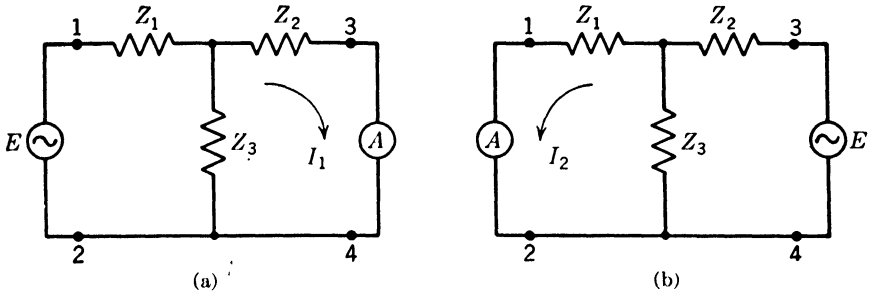


FIG. 15.6. Circuits to illustrate the reciprocity theorem.

the ammeter A when the generator is producing a voltage E at the terminals 1 and 2 is given by

$$I_1 = \frac{EZ_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \tag{15.22}$$

If now the generator and ammeter are interchanged, as in Fig. 15.6*b*, without otherwise changing the network, the current I_2 that will flow through the ammeter A will be given by

$$I_2 = \frac{FZ_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3} \tag{15.23}$$

which is seen to be identical with Eq. (15.22), and which therefore proves the theorem.

The reciprocity theorem has many important uses in other applications besides electrical network theory. It can be shown to hold in any situation where the elements obey Ohm's law, that is, where the superposition theorem applies. Thus, for example, in the transmission and reception of radio signals where two antennas, one transmitting and the other receiving, have a certain relationship, this same relationship can, by the reciprocity theorem, be shown to hold when their functions are reversed, that is, when the transmitting antenna becomes a receiving antenna, and vice versa. The same relationship holds in acoustics, and the reciprocity principle is widely used in the absolute calibration of microphones and loud-speakers. A microphone that is picking up a signal from a loud-speaker can itself be connected as a loud-speaker, and the loud-speaker be made to serve as a microphone. The resulting measurements can be used to get a reading of the absolute level of each.

15.7. Maximum-power-transfer Theorems. There is a well-known theorem, which the student encounters early in the study of d-c electric-

ity, to the effect that a generator will deliver the maximum power to a load when the resistance of the load equals the internal resistance of the generator. This theorem is illustrated in Fig. 15.7, which shows the power delivered to a variable load by a d-c voltage generator of 1 ohm internal resistance.

This theorem has several counterparts in a-c theory, two of which will now be discussed.

An a-c generator will deliver the maximum power to a load when the impedance of the load is the conjugate of the internal impedance of the generator.

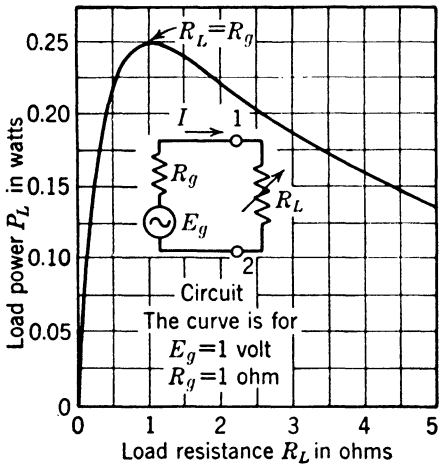


FIG. 15.7. Generator with internal resistance R_g delivering power to the load resistance R_L as it is varied.

Note specifically in this theorem that it is the *power* that is delivered to the receiving impedance which is a maximum, and not merely the volt-ampere product. *Power* can be absorbed only by a resistance, as was previously shown in Chap. 10 (page 171). No matter what the current through a pure reactive element may be, the total power drawn over a complete cycle is zero. A reactive element will draw power over a portion of a cycle, but the power is stored and returned to the generator over the remainder of the cycle. Thus, if the receiving impedance in the theorem is given by

$$Z_r = R_r + jX_r \tag{15.24}$$

and the current through this impedance equals I, then the *power* P_r absorbed by, or delivered to, Z_r will be given by

$$P_r = |I|^2 R_r \tag{15.25}$$

irrespective of the value of X_r.

Let the impedance Z_s of the generator be given by

$$Z_s = R_s + jX_s \tag{15.26}$$

Then for any value of R_r, the condition that will make the current I around the circuit a maximum is that

$$X_r = -X_s \tag{15.27}$$

It is easy to see that this is true, since when this relationship holds, the circuit is in series resonance. Hence, if R_r is fixed, the maximum power that will be delivered to the receiving impedance will be given when Eq. (15.27) holds. But when this equation is true, the reactive components of the load and the generator can be neglected, since they neutralize each other, and the same current flows as if they were both zero. Hence it is only necessary to find the value of R_r that would draw the maximum power from the generator if it had an internal impedance equal to R_s alone. This, of course, is given immediately by

$$R_r = R_s \tag{15.28}$$

since if reactances are omitted, the laws for a-c circuits are the same as the laws for d-c circuits. Hence we can write at once that the condition that a complex impedance $R_r + jX_r$ will draw maximum power from an a-c generator of complex internal impedance $R_s - jX_s$ is that

$$R_r + jX_r \text{ equals conjugate of } R_s - jX_s \tag{15.29}$$

In other words, maximum power will be delivered by an a-c generator when the load impedance is the conjugate of the generator impedance.

Example 14.5. An a-c generator delivers 10 volts rms and has an internal impedance of $50 + j10$ ohms. What is the maximum power that can be delivered by this generator?

Solution. By the theorem just given, the maximum power will be delivered to a load of $50 - j10$ ohms. Under these conditions the total current around the circuit will be given by

$$I = \frac{10}{50 + j10 + 50 - j10} = \frac{10}{100} = 0.1 \text{ amp rms}$$

The total power delivered to the load, which is the maximum power that the generator can deliver, will be given by

$$(0.1)^2(50) = 0.5 \text{ watts} \qquad \textit{Ans.}$$

It may often happen that both the generator and load impedances are fixed. Under such conditions it will, of course, be impossible to deliver the maximum power to the load. It may, however, be permissible to insert a transformer between the load and the generator. An ideal transformer will vary the *magnitude* of the load as seen from the transformer terminals, but will not vary its phase angle, that is, the proportion of resistance to reactance in the load. It is often a matter of importance to find the value of impedance that will draw the maximum amount of power from the generator under the condition that the ratio of resistance to reactance is given, but that the absolute magnitude of the impedance

can be varied. Under these conditions it can be shown that the following theorem holds:

If the ratio of the resistance to the reactance of a given impedance is constant, but if the absolute magnitude of the impedance can vary, then the impedance will draw the maximum power from a generator when its absolute value equals the absolute value of the impedance of the generator.

Let

$$Z_s = R_s + jX_s = R_s(1 + jm) \tag{15.30}$$

and

$$Z_r = R_r + jX_r = R_r(1 + jn) \tag{15.31}$$

Then

$$I = \frac{E}{R_s(1 + jm) + R_r(1 + jn)} \tag{15.32}$$

and

$$|I|^2 = \frac{E^2}{(R_s + R_r)^2 + (R_s m + R_r n)^2} \tag{15.33}$$

and

$$P_r = |I|^2 R_r = \frac{E^2 R_r}{(R_s + R_r)^2 + (R_s m + R_r n)^2}$$

$$= \frac{E^2}{\dfrac{(R_s + R_r)^2}{R_r} + \dfrac{(R_s m + R_r n)^2}{R_r}} \tag{15.34}$$

Since E^2 is a constant, it follows that if P_r is to be a maximum, the denominator in Eq. (15.34) must be a minimum. Let this denominator equal y. Then

$$y = \frac{(R_s + R_r)^2}{R_r} + \frac{(R_s m + R_r n)^2}{R_r}$$

$$= \frac{R_s^2}{R_r} + 2R_s + R_r + \frac{R_s^2 m^2}{R_r} + 2R_s mn + R_r n^2 \tag{15.35}$$

By the calculus, if y has a minimum, dy/dR_r must equal zero at this minimum. Differentiating Eq. (15.35) and setting it equal to zero, remembering that R_s, m, and n are constant, we have

$$\frac{dy}{dR_r} = -\frac{R_s^2}{R_r^2} + 1 - \frac{R_s^2 m^2}{R_r^2} + n^2 = 0$$

or

$$\frac{R_s^2}{R_r^2} + \frac{R_s^2 m^2}{R_r^2} = 1 + n^2$$

or

$$R_s^2(1 + m^2) = R_r^2(1 + n^2) \tag{15.36}$$

In other words, the condition that the power delivered to the load should be a maximum is that the absolute values of the generator and load impedances should be equal.

Summary

Superposition Theorem. If a voltage E_1 in series with branch A_1 in a network causes a current I_1 to flow through branch B_1 of the network, and if a voltage E_2 in branch A_2 (which may be identical with A_1) of the network causes a current I_2 to flow through branch B_1 of the network, then if voltage E_1 in branch A_1 and voltage E_2 in branch A_2 operate simultaneously, the current that will flow through branch B_1 will equal $I_1 + I_2$.

Thévenin's Theorem. With respect to a given pair of terminals, any network containing any number of generators can be replaced by a single generator in series with a single impedance connected across the given terminals. The voltage of the generator is given by the open-circuit voltage that would be measured across the terminals in the absence of a load, and the impedance is given by the impedance that is seen looking back into the terminals when all generators are replaced by their internal impedances.

Norton's Theorem. With respect to any pair of output terminals, any network containing any number of generators and impedances can be replaced by a single constant-current generator in parallel with a single impedance connected across the given output terminals. The current produced by the generator is equal to the current that would flow between the output terminals if they were short-circuited, and the internal impedance of the equivalent generator is given by the impedance that is seen looking back into the output terminals when all generators in the circuit are replaced by their internal impedances.

Reciprocity Theorem. If a voltage E in a branch A of a network causes a current I to flow in another branch B of the network, then if the same voltage E is introduced in the branch B of the network, the same current I will flow in branch A.

Maximum-power-transfer Theorem. *a.* An a-c generator will deliver the maximum power to a load when the impedance of the load is the conjugate of the internal impedance of the generator.

b. If the ratio of the resistance to the reactance of a given impedance is constant, but the absolute magnitude of the impedance can vary, then the impedance will draw the maximum power from a generator when its absolute value equals the absolute value of the impedance of the generator.

Exercises

15.1. Find I in Fig. 15.8 by means of Thévenin's theorem. Draw the equivalent circuit.

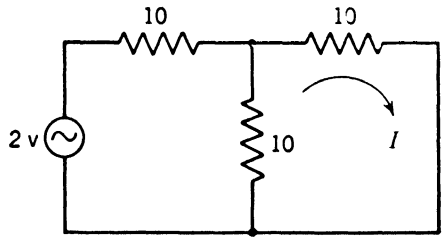


FIG. 15.8. Circuit for Exercise 15.1.

15.2. Find I in Fig. 15.8 by means of Norton's theorem. Draw the equivalent circuit.

15.3. Find I in Fig. 15.9 by means of Thévenin's theorem.

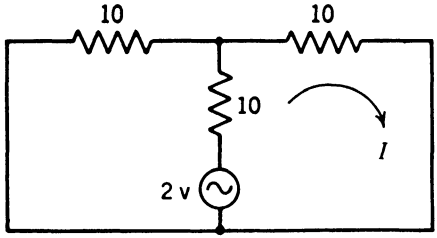


FIG. 15.9. Circuit for Exercise 15.3.

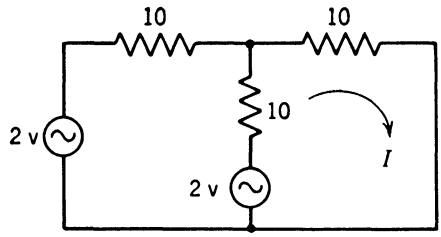


FIG. 15.10. Circuit for Exercise 15.4.

15.4. Find I in Fig. 15.10 by means of the superposition theorem.

15.5. Find the current through the galvanometer G (of 1,000 ohms internal resistance) in Fig. 15.11 by means of Thévenin's theorem. Draw the equivalent circuit.

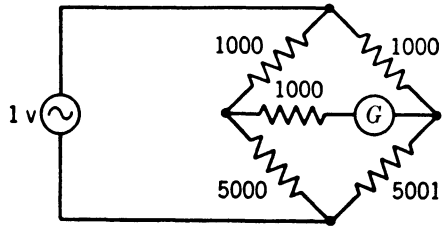


FIG. 15.11. Circuit for Exercise 15.5.

15.6. What is the maximum power that can be delivered to a load by a 1-kv generator of $10 + j0.8$ ohms internal impedance?

15.7. Into what load will the generator of Exercise 15.6 deliver the maximum power?

15.8. If the generator of Exercise 15.6 is to be matched to a load whose phase angle is $+45°$, what should this load be to absorb the maximum power?

15.9. If a generator and ammeter are connected as shown in Fig. 15.12, and a current of 10 amp flows through the ammeter, what current will flow through the ammeter when it is interchanged with the generator?

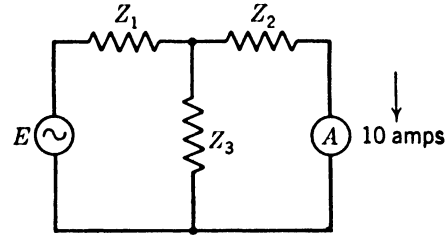


FIG. 15.12. Circuit for Exercise 15.9.

15.10. Under what circumstances does the superposition theorem apply to a network?

CHAPTER 16

THERMIONIC EMISSION

All the circuit elements so far considered, with the exception of generators, have been "passive" elements. By a *passive* element is meant one that contributes no power to a circuit, and that requires no external sources of power for its operation. Resistors, capacitors, and inductances are typical passive circuit elements.

There is another large and important class of circuit elements that are classified as *nonpassive*, or *active*, elements. These may or may not contribute power to a circuit, but all require external sources of power for their operation. The most important of these, from the standpoint of the communications engineer, are the various forms of electron tubes, which comprise the subject of this and the succeeding chapters of this book. Considered purely as a circuit element, an electron tube is a device through which a current is made to pass, whose resistance can be made to vary by the application of external potentials to a control element in the tube. The great value of such a circuit element lies in the fact that little or no power need be consumed in the control circuit, while at the same time there is no limit to the power that can be controlled by the tube.

Only the basic physics of electron-tube behavior can be discussed here. The applications of electron tubes constitute a broad field in themselves, and will be considered in greater detail in another volume of this series.

16.1. Introduction. The primary duty of any electron tube is to produce, control, and collect a stream of electrons. Since electrons are all exactly alike, it does not matter from what source they are obtained. The principal sources of electrons result from

1. Thermal emission (thermionic devices)
2. Electrostatic fields (discharge tubes)
3. Secondary emission (electron multiplier tubes)
4. Photoelectric effect (photoelectric cells)
5. Spontaneous disintegration (radioactive substances)

This chapter will be devoted to a study of thermionic emission because the majority of electron tubes depend upon thermal agitation in a hot cathode for their supply of electrons. First, however, a brief historical sketch will be given, followed by a quantitative treatment of electron current and voltage.

16.2. Edison Effect. Thomas A. Edison in 1883 discovered, while endeavoring to produce a better carbon-filament lamp, that connecting

a plate from the inside of the tube to the positive terminal of the filament caused a weak current to flow to the plate but that no current flowed when the plate was connected to the negative end of the filament. This phenomenon has since been called the *Edison effect* and is perhaps the real starting point of the thermionic tube. As we now know, the electrons emitted from the hot filament were attracted to the plate when it was connected to the positive end of the filament. This flow of electrons constitutes an electron current. When the connection was made to the negative terminal, the electrons were repelled; hence no electron current flow was observed.

16.3. Electron Current. An electron in motion is the smallest natural element of an electric current. It takes 6.285×10^{18} electrons passing a point every second to constitute 1 amp of current. A given current may be composed of relatively few electrons moving rapidly or many electrons drifting along slowly. An electron with a charge e moving at a velocity v is equivalent to a current of magnitude ev, and if there are n electrons, the current measured in electrons per second is

$$i = nev \tag{16.1}$$

where i = current, amp
n = number of free electrons per centimeter length of conductor
$e = 1.591 \times 10^{-19}$ coulomb = charge on 1 electron
v = average velocity of electrons, cm/sec

This equation can be considered as giving the current along a conductor. Since n is the number of free electrons in the conductor per unit length, the area does not have to be mentioned. The *space-current density* in an electron tube, for example, can be expressed by Eq. (16.1) if the terms are defined as follows:

i = current density, amp/sq cm
n = electron density, free electrons/cu cm
$e = 1.591 \times 10^{-19}$ coulomb = charge on 1 electron
v = average velocity of electron drift, cm/sec

Example 16.1. A No. 14 gauge copper wire carries a 20-amp current. If the area of the conductor is 2.081 sq mm, determine the average velocity of electron drift. Assume 1 free electron per atom, which amounts to 8.47×10^{23} free electrons per cubic centimeter of copper.

Solution. The current density is

$$i = \frac{\text{amperes}}{\text{sq cm area}} = \frac{10}{0.02081} = 480.5 \text{ amp/sq cm}$$

By Eq. (16.1) the average electron drift velocity is only

$$v = \frac{i}{ne} = \frac{480.5}{(8.47 \times 10^{23})(1.591 \times 10^{-19})} = 3.565 \times 10^{-3} \text{ cm/sec} \qquad \textit{Ans.}$$

16.4. Electron Voltage. If electrons in free space are acted on by a constant force, as shown in Fig. 16.1, they will gain velocity, owing to a constant acceleration, according to Newton's second law of motion.

$$F = ma \tag{16.2}$$

where F = force of attraction, dynes
m = mass, g
a = acceleration, cm/sec^2

$$a = \frac{v - v_0}{t} \tag{16.3}$$

$$v = \frac{2s}{t} \tag{16.4}$$

where v = final velocity, cm/sec
v_0 = initial velocity, cm/sec
$\bar{v} = s/t$ average velocity, cm/sec
s = distance traveled, cm
t = time, sec

The increase in velocity will increase the spacing between electrons, and

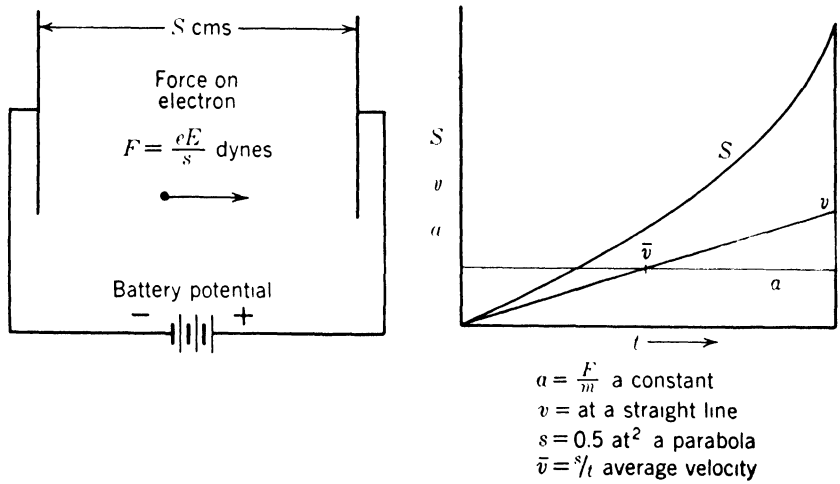


FIG. 16.1. An electron in a uniform electrostatic field.

hence will decrease the number of electrons per unit volume, if the current, as given in Eq. (16.1), is to remain constant. The voltage gradient (voltage drop per unit length), however, will increase as the electron velocity increases. The term *electron-volt,* commonly used to express this electron energy, is defined as *the energy gained by an electron when accelerated in free space through a difference of potential of one volt.* The energy necessary to excite or ionize atoms is commonly expressed in terms of electron-volts.

The potential energy gained by an electron if it is moved from the positive to the negative plate, as shown in Fig. 16.1, is

$$\text{Potential energy} = eE \tag{16.5}$$

where the potential energy is that of 1 electron, in ergs, and

$e = 1.591 \times 10^{-20}$ abcoulomb = charge on the electron
= (coulombs)(0.1)

E = potential between the two plates, abvolts
= (volts)(10^8)

Now if the electron is allowed to start from rest at the negative plate, it will gain kinetic energy, owing to its velocity, as given by

$$\text{Kinetic energy} = 0.5mv^2 \tag{16.6}$$

where the kinetic energy is that of 1 electron, in ergs, and

$m = 9 \times 10^{-28}$ g = mass of 1 electron

v = velocity of electron, cm/sec

But the potential energy of the electron at the negative plate is exactly equal to its kinetic energy at the positive plate; hence Eqs. (16.5) and (16.6) can be equated to get

$$eE = 0.5mv^2 \tag{16.7}$$

The force on the electron can also be written in terms of the potential energy and the distance; thus

$$F = \frac{eE}{s} \tag{16.8}$$

This equation can be derived from the preceding equations.

Example 16.2. An electron starting from rest at the cathode is acted on by 1 volt applied to a plate 0.5 cm from the cathode. Determine the acceleration of the electron, and its velocity and energy when it strikes the plate. How long does it take to make the transit, and what is the value of the force causing it to move?

Solution. By Eq. (16.5) the energy of the electron is

$$\text{Potential energy} = eE = (1.591 \times 10^{-20})(10^8) = 1.591 \times 10^{-12} \text{ erg} \qquad \textit{Ans.}$$

(This is the ergs of energy in 1 electron-volt.) Knowing the mass, we have from Eq. (16.7)

$$1.591 \times 10^{-12} = (0.5)(9 \times 10^{-28})v^2$$

$$v = 5.946 \times 10^7 \text{ cm/sec} \qquad \textit{Ans.}$$

$$= \frac{5.946 \times 10^7}{1.609 \times 10^5 \text{ cm/mile}} = 369.6 \text{ miles/sec}$$

By Eq. (16.4) the time of transit is

$$t = \frac{2s}{v} = \frac{(2)(0.5)}{5.946 \times 10^7} = 1.682 \times 10^{-8} \text{ sec} \qquad \textit{Ans.}$$

By Eq. (16.3) the acceleration is

$$a = \frac{v}{t} = \frac{5.946 \times 10^7}{1.682 \times 10^{-8}} = 3.535 \times 10^{15} \text{ cm/sec}^2 \qquad \textit{Ans.}$$

Finally, by Eq. (16.2)

$$F = ma = (9 \times 10^{-28})(3.535 \times 10^{15}) = 3.182 \times 10^{-12} \text{ dyne} \qquad \textit{Ans.}$$

Or by Eq. (16.8)

$$F = \frac{eE}{s} = \frac{1.591 \times 10^{-12}}{0.5} = 3.182 \times 10^{-12} \text{ dyne} \qquad \textit{Check}$$

16.5. Thermionic Emission. At absolute zero (273° below 0°C), atomic and electronic motion is supposed to cease. It is, therefore, the lowest possible temperature. Temperatures measured from this point, as zero, are known as *absolute temperatures.* As the temperature of a solid increases, the activity of the electrons and atoms increases to cause expansion of the solid. Therefore, the spacing between the atoms and electrons is increased, which reduces the forces of attraction holding the solid in shape. As the temperature is increased, the forces of attraction will be overcome, and the solid will change into a liquid. If the atoms and electrons have sufficient energy, they will break down the surface tension and change into a vapor.

The evaporation of electrons from the surface of a hot solid is known as *thermionic emission.* This can be accomplished without changing the solid into a liquid if a metal containing heavy atoms and having a high melting point is used. If the temperature of the solid is high enough, the velocities acquired by some of these free electrons are sufficient to project them outside the sphere of attraction of the solid. To do this, the kinetic energy of the free electron must exceed the work necessary for an electron to overcome the surface tension of the solid.

The energy lost by a unit charge when passing through the surface of a substance is known as the *work function.* The work function has different values for different metals. It varies roughly from 10^{-12} to 10^{-11} erg or from 0.625 to 6.25 electron-volts. One might assume that the metal with the lowest work function would be the best emitter, but since the emitting surface must possess certain mechanical features, such as mechanical strength at the operating temperature, this is not necessarily true.

16.6. Pure-tungsten Emitters. Since tungsten has the highest melting point of all metals, 3350°K (absolute temperature), and is composed of atoms 184 times heavier than the simple hydrogen atom, it is well suited for thermionic emission. Furthermore, it can be subjected to the extremely high temperatures that are necessary for proper evacuation of the tubes. Since tungsten is very brittle, it must be worked while hot into a fibrous condition so that it can be formed into the proper shape for use in filaments.

Tungsten emitters are particularly desirable for high-power operation where durability under exacting conditions is important. Tungsten

filaments, if allowed to burn without plate voltage, have the property of absorbing the gas in the tube, thus preventing gas ionization, which cannot be tolerated in high-power operation. Tungsten filaments have a high power consumption and a comparatively short life owing to the high operating temperatures required to give good emission.

Example 16.3. If a hydrogen atom is 1,847 times heavier than an electron, how much heavier is a tungsten atom than an electron?

Solution. The weight of a tungsten atom is (1,875)(184) = 343,848 times heavier than the weight of an electron. *Ans.*

16.7. Thoriated-tungsten Emitters. Impurities in metals may cause marked changes in the work function. These impurities usually collect at the surface and form an atomic film over the surface. If the atomic film is electropositive, as an oxygen film is, the work function is increased; however, if the atomic film is electronegative, like that of thorium, the work function is decreased. Langmuir and Rogers discovered that a small amount of thorium in tungsten materially increased the emission. Thoriated-tungsten emitters are now made by dissolving a small amount of thorium oxide and carbon in a tungsten emitter. After evacuation in a tube, the emitter is activated by *flashing* at a high temperature to reduce some of the thorium oxide to pure thorium. The emitter is then *aged* by operating for some time at a relatively high temperature to permit the pure thorium to work its way to the surface to form a layer one atom thick. If the emitter is operated at the proper temperature, the atomic layer of pure thorium is replenished from within the emitter as rapidly as evaporation takes place at the surface. If the temperature is too high, the thorium will boil out of the emitter to reduce the supply of pure thorium. If the operating temperature is too low, the atomic surface layer will evaporate without being replenished. The pure thorium can be brought to the surface by operating the tube for a while above operating temperature. From this we can conclude that it is best to *operate such emitters as specified by the manufacturer.*

An emitter depleted of pure thorium atoms on the surface will have very low emission. Such tubes can usually be *rejuvenated* several times by flashing the tube and aging it, just as the emitter is prepared when manufactured. In this process, thorium oxide in the cathode is reduced to pure thorium, and certain impurities such as gas atoms may be driven out of the cathode. During the aging process, the temperature can be reduced to normal several times to check the emission current. If the emission does not come up to normal, the thorium is nearly exhausted, which indicates the end of useful tube life.

For medium- and low-power work, thoriated tungsten tubes can be

sufficiently well evacuated for excellent operation. The filament-type tube can be made with a very small diameter filament to permit low power consumption, and has a long life due to the low operating temperature.

16.8. Carbonized Thorium-tungsten Emitters. *Carbonization* of a thoriated-tungsten emitter protects it against positive-ion bombardment, reduces thorium evaporation by approximately 85 per cent, and permits the use of higher operating temperatures. The carbonization is accomplished by operating the thoriated-tungsten emitter in an atmosphere of some hydrocarbon, the molecules of which combine with the tungsten to form a surface of brittle tungsten carbide. These emitters are replacing many of the pure-tungsten emitters for more efficient highpower operation.

16.9. Oxide-coated Emitters. Oxide-coated emitters are usually made by coating a nickel alloy with barium and strontium carbonates. During the process of evacuation, the emitter is heated above operating temperature to oxidize the carbonates. Then at normal emitter operating temperature, anode voltage is applied, until normal emission current is reached, which seems to draw the pure metallic barium and strontium to the surface of the oxide coating. During normal operating conditions, the evaporated outer layer is replenished by diffusion of pure metals from the interior of the oxide coating.

Oxide-coated emitters give high electron emission at very low temperatures with comparatively long life. It is difficult to maintain a good vacuum in this type of tube because of the porous nature of the coating, which holds gas. The emission is reduced in the presence of gas owing to oxidation of the active material or coating of this material by positive-ion bombardment. The emitting surface can also be destroyed by hot spots where the emission is high; owing to the high current at these spots, the temperature increases sufficiently to remove the coating. Since hot spots are most likely to occur at high voltages, these emitters are most widely used in small high-vacuum tubes. The power consumption is slightly more than that necessary for the thoriated-tungsten emitter.

16.10. Getters. For vacuum tubes containing emitters other than pure tungsten, *getters* are usually used to remove the residual gas in the tube. Getters are made of active chemical substances, such as magnesium, which have the property of combining with the gases when they are vaporized. In glass tubes, the getter is placed in a small cup inside the tube and flashed after the tube has been evacuated. The released vapors will combine with the gases released from an oxide-coated emitter, the electrodes, and the glass wall of the tube. In the metal tubes, the shell is connected to the base pin through a helical coil that is coated with the getter. After the tube has been evacuated, a current passing through the coil vaporizes the getter to take up the residual gases.

16.11. Filamentary and Indirectly Heated Cathodes. Thermionic emitters can be divided into two classes: *filamentary* and *indirectly heated.* The earliest vacuum tubes contained only filamentary emitters, which were heated by passing current through them. The advent of a-c sets led to the development of indirectly heated emitters.

The filamentary-type emitter, if heated with direct current, has an IR drop from one end to the other equal to the filament voltage. This difference in potential with respect to the plate (or grid) makes the space current vary along the filament, which in turn makes the filament current and temperature vary along the total length of the filament. Since the space current increases the electron flow at the negative end, the temperature will be increased to cause more emission. Furthermore, the emitted electrons cause additional heating at the surface with the result that hot spots may develop owing to the excessive temperature. The varying filament-to-plate potential rounds off the knee of the space-charge saturation curves as shown in the next chapter.

Large transmitters using direct current for the filaments usually have a filament-current reversing switch, so that the polarity can be reversed from day to day to cause both ends of the filament to carry this additional current for equal periods of time. This greatly prolongs the useful life of the tube.

Usually alternating current is not applied to filamentary-type emitters, because of the variation in voltage and temperature along the filament. This results in a variation of emission current, which introduces a hum component into the space current of the tube that is, in most cases, objectionable.

The indirectly heated emitters or cathodes have a uniform potential and temperature with electrical isolation from the heater element. This permits the use of alternating current in the heater and the operation of the cathode at some other potential. Most indirectly heated cathodes, as used in receivers, are oxide coated because of the low operating temperature and the sluggishness of the oxide coating in responding to the varying heater temperature when alternating current is used. Indirectly heated emitters are more sluggish than filamentary emitters because of the time necessary for the heat transfer from the heater to the cathode. Emission from the heater must be shielded from the plate and is kept from the cathode by operating the cathode at a negative potential with respect to the heater.

Summary

Electrons are emitted from the surface of many materials when the temperature is sufficiently high. This phenomenon is known as *thermionic emission.*

The energy lost by a unit charge when passing through the surface of a substance is known as the *work function.*

The energy gained by an electron when accelerated in free space through a potential of 1 volt is called 1 electron-volt.

Tungsten has the highest melting point of any metal, 3350°K. It is widely used for the filaments of electron tubes, particularly in high-power tubes.

Thoriated-tungsten filaments are made by dissolving a small amount of thorium oxide and carbon in a tungsten emitter. Thoriated-tungsten filaments emit many more electrons than a pure-tungsten filament for the same operating power, but cannot be operated at as high a temperature.

An oxide-coated filament is usually made by coating a nickel alloy with barium and strontium carbonates. An oxide-coated emitter is much more efficient than either a tungsten or a thoriated-tungsten filament, but must be operated at a much lower temperature.

A getter is a chemical substance that unites with the residual gas in an electron tube after the tube has been pumped and sealed.

A filamentary-type emitter is one in which electrons are emitted directly from the wire that carries the heating current.

An indirectly heated emitter is one in which there is no direct connection between the wires that carry the heating current and the emitting surface.

Exercises

16.1. What is the Edison effect?

16.2. How many electrons flow through the filament of a 100-watt incandescent lamp in 1 yr if the applied voltage is 100 volts? If the filament weighs 0.02 g, how long must the current flow to make the combined mass of the electrons equal to the weight of the filament? (One common year = 365 days.)

16.3. If 250 volts is applied to the anode of an electron tube, which is spaced 0 3 cm from the cathode, determine the force exerted on a free electron if the field is uniform (assuming large parallel plates for the cathode and anode). If the electron starts from rest at the cathode, what is its velocity when it reaches the plate, and how long does it take to get there?

16.4. How many electrons per centimeter would it take in Exercise 16.3 to make an average current flow of 1 amp? (Assume no reaction between the electrons themselves.)

16.5. Define the work function of a metal. How does it affect the thermonic emission?

16.6. Why is it important to operate emitters as specified by the manufacturer?

16.7. What kind of tubes can be rejuvenated? How is rejuvenation accomplished?

16.8. Explain the construction, merits, and use of a pure-tungsten emitter.

16.9. Explain the construction, merits, and use of a carbonized thoriated-tungsten emitter.

16.10. Give three reasons for the occurrence of hot spots on the negative end of a filamentary-type emitter.

CHAPTER 17

THE DIODE TUBE

In Chaps. 1 and 2 we studied some of the properties of an electron, while in Chap. 16 we treated thermionic emission as a method of producing a supply of electrons. We are now ready to study the action of the electrons after they are emitted and the means by which they can be collected by a plate.

The two-element tube commonly known as a *diode* consists essentially of a *cathode* and an *anode*. Because of the shapes of the electrodes, the cathode is commonly called the *filament* and the anode the *plate*. The cathode is usually hot, having a low affinity for electrons, and the anode cold, having a positive voltage with respect to the cathode. The two elements are mounted in an evacuated vessel.

The diode tube is valuable because of its unilateral conductivity. This means that the electron stream can flow only in one direction, from the cathode to the anode. There are a number of devices that have this property—the crystal detector, copper oxide rectifier, selenium rectifier, mercury rectifier, Tungar rectifier, gas-filled tube rectifiers, and the diode or kenotron rectifiers. The diode or kenotron that operates as a thermionic valve will be treated in this chapter.

17.1. Cathode Emission. The universally adopted equation for the cathode emission is a revision of an equation originally proposed by Richardson. The total electron emission current per unit area is given by the equation[1]

$$I_S = AT^2\epsilon^{-b_0/T} \tag{17.1}$$

where I_S = total electron emission current, amp/sq cm
A = 60.2, a constant for all pure metals
= 3 for thoriated tungsten
= 10^{-2} for barium-strontium oxides
T = absolute temperature, °K
ϵ = 2.71828 = the Napierian base of natural logarithms

[1] The symbols used are those adopted by the Institute of Radio Engineers in so far as they apply. See "Standards on Electronics," definitions of terms, symbols, 1938 (reprinted 1943) and "Standards on Electronics, Methods of Testing Vacuum Tubes," 1938 (reprinted 1943). See Appendix.

$b_0 = \frac{\phi_0}{K}$ = a constant depending upon the work necessary to move an electron from the inside to the outside of the emitter

ϕ_0 = work function measured in electron-volts

$K = 8.63 \times 10^{-5}$ = Boltzmann's constant, volts/°K

This equation of the total emission or saturation current is a function of the temperature only; therefore, it is assumed that all electrons emitted are drawn away by an intense electric field. Under these conditions the emission current I_s of Eq. (17.1) when plotted against the temperature T of the emitter will give a curve as shown in Fig. 17.1.

Example 17.1. If the work function of pure tungsten is 4.52 electron-volts, determine the total emission current in amperes from a filament having 0.2 sq cm area and operating at a temperature of 2600°K.

Solution. In Eq. (17.1)

$$-b_0 = -\frac{\phi_0}{K} = \frac{-4.52}{8.63 \times 10^{-5}} = -52{,}400$$

Substituting in Eq. 17.1

$$I_S = (60.2)(2{,}600)^2(2.71828)^{-52{,}400/2{,}600}$$

$$= (4.07 \times 10)^8(2.71828)^{-2\,30} = \frac{4.07 \times 10^8}{6.547 \times 10^8}$$

$$= 0.622 \text{ amp/sq cm}$$

The total emission is then

$$(0.2)(0.622) = 0.1244 \text{ amp} \qquad \textit{Ans.}$$

17.2. Space Charge. The curve of Fig. 17.1 is made on the assumption that all the electrons are drawn away from the emitter and that hence there is no space charge. If an emitter is placed in an evacuated vessel,

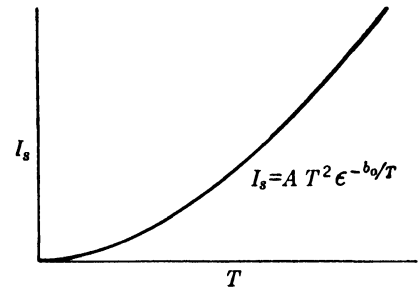


FIG. 17.1. Emission current I as a function of temperature T ($E_p = \infty$).

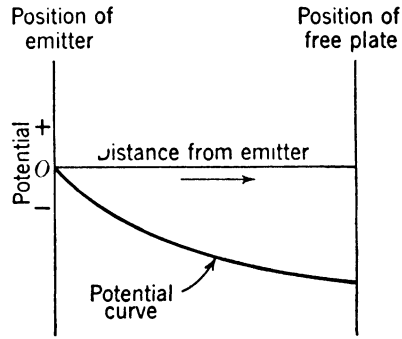


FIG. 17.2. Potential distribution about a cathode with a free plate.

the electrons emitted will form a *negative space charge* around the emitter. This negative space charge causes emitted electrons to slow down as they travel from the emitter; hence the space charge composed of these electrons will continue to exist, giving a potential distribution as shown in Fig. 17.2.

This curve shows that the potential becomes more negative as the distance from the filament is increased, until the electron space charge becomes uniform, giving a constant negative potential. If a free plate is present in the evacuated vessel, it will assume this negative potential as shown in the figure. This is called the *Edison effect* mentioned on page 265.

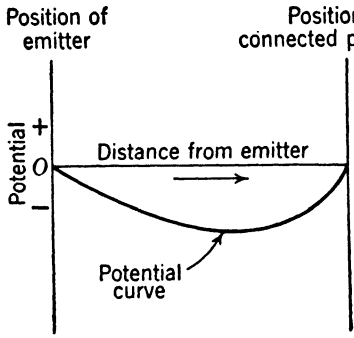


FIG. 17.3. Potential distribution between a cathode and plate which is connected to the cathode.

If the plate is connected to the emitter, it will stay at the same potential with the negative space charge existing between the emitter and plate. The potential distribution for this case is shown in Fig. 17.3.

17.3. Child's Voltage Law. Consider the case of two large parallel plates in an evacuated vessel, one the cathode and the other the anode. With a positive voltage on the anode and a *cold cathode*, the potential distribution will be uniform, giving a straight line as shown in Fig. 17.4.

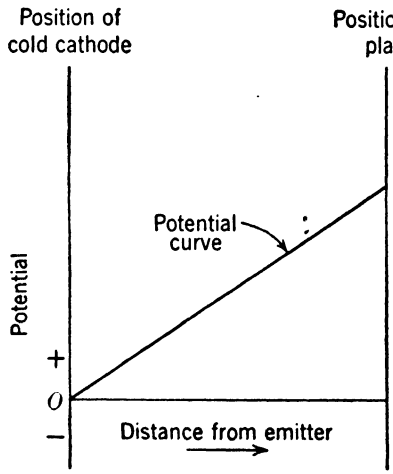


FIG. 17.4. Potential distribution between a cold cathode and plate.

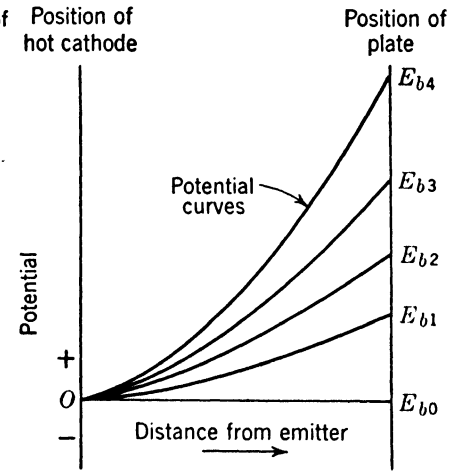


FIG. 17.5. Potential distribution between a hot cathode and plate for various plate voltages when the initial electron velocity is zero.

Now if the cathode emits electrons from its surface with *zero initial velocity*, the potential distribution for a positive plate will be as shown in Fig. 17.5. Potential distributions have been plotted for several anode voltages. If corresponding plate currents are plotted for these voltages, the curve of Fig. 17.6 will result. The equation of this curve was first

derived by Child and for parallel plates is

$$I_b = \frac{A(2.34 \times 10^{-6})E_b^{3/2}}{S^2} \tag{17.2}$$

where I_b = plate or anode current, amp
A = area of the plate, sq cm
E_b = plate or anode voltage, volts
S = spacing between the cathode and anode, cm

The above equation for the diode current is quite accurate in actual practice if the voltage between the anode and cathode is large in comparison with the voltage drop along the emitter and the initial velocities of emission. The plate voltage must be well below saturation, as described on page 274, and the number of electrons emitted from the cathode must be large.

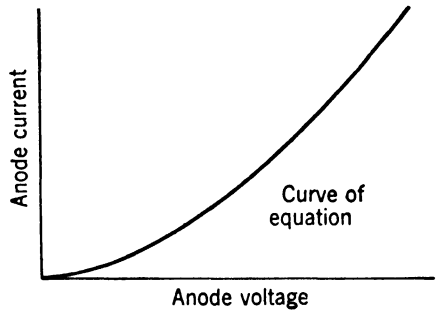


FIG. 17.6. Characteristic curve of a diode for low-plate voltages.

Example 17.2. Determine the plate current in amperes in a tube having a plate area of 1 sq cm and spaced 0.5 cm from the cathode, with a voltage of 100 volts.

Solution. Substituting in Eq. (17.2)

$$I_b = \frac{(1)(2.34 \times 10^{-6})(100)^{3/2}}{(0.5)^2}$$
$$= (9.36 \times 10^{-6})(100)^{3/2} = 9.36 \times 10^{-3} \text{ amp} \qquad \textit{Ans.}$$

An electrode assembly consisting of two parallel plates in many cases is not practical. The usual procedure is to make the electrodes in the form of coaxial cylinders. For this case, Langmuir has developed the following equation:

$$I_b = \frac{l(14.68 \times 10^{-6})E_b^{3/2}}{r\beta^2} \tag{17.3}$$

where I_b = plate or anode current, amp
l = length of the cylinders, cm
E_b = plate voltage, volts
r = radius of the plate, cm
β^2 = 1, which gives an accuracy within a few per cent when the ratio of the diameter of the anode to the diameter of the cathode is greater than 100

In this equation it should be noted that the current depends upon the first power of the radius; hence the spacing can be increased to reduce

electrode capacities without reducing the plate current as rapidly as in the case of Eq. (17.2) for parallel electrodes. This is a valuable consideration in the design of vacuum tubes for high-frequency use.

Example 17.3. Determine the plate current in a diode having a plate voltage of 400 volts applied to the concentric plate 2 cm in diameter and 3 cm in length. Assume the cathode to be a very small cylinder at the center.

Solution. Substituting in Eq. (17.3) gives

$$I_b = \frac{(3)(14.68 \times 10^{-6})(400)^{3/2}}{(1)(1)} = 352.5 \text{ ma} \qquad \textit{Ans.}$$

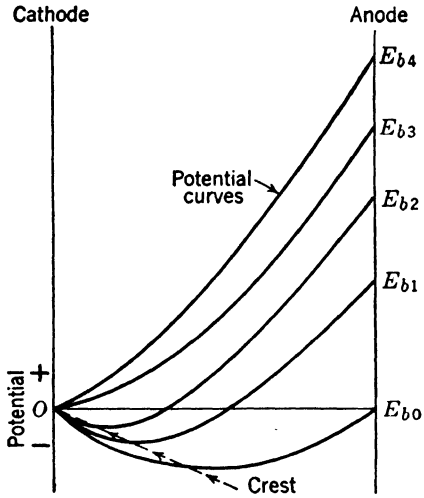


FIG. 17.7. Potential distribution between a hot cathode and an anode showing the effect of space charge for various anode voltages when the initial electron velocity is not zero.

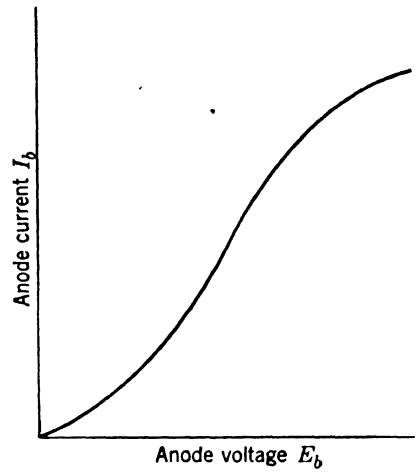


FIG. 17.8. $E_b - I_b$ characteristic curve for a diode.

If the effect of *initial velocities* is introduced, the curves of Fig. 17.5 are changed to those of Fig. 17.7. In this case, there is a space charge near the emitter for the lower anode voltages. As the plate voltage is increased, the space charge is diminished, because the electrons are drawn to the plate more rapidly, until the crest (point of maximum negative potential) reaches the surface of the emitter. For this voltage all electrons leaving the emitter will travel to the plate. If the voltage is still further increased, the current will not increase proportionally because free electrons are not available. Figure 17.8 shows this saturation effect for the higher plate voltages.

17.4. Temperature and Space-charge Saturation. Since both temperature of the emitter and plate voltage determine the plate current, a graph showing the relationship among these three variables will be three-

dimensional, or, in other words, will be a surface instead of a curve. Figure 17.9 shows such a surface. The various curves are obtained from intersections of the surface with planes perpendicular to one axis. For instance, the curve of Fig. 17.8 is obtained by the intersection of a plane perpendicular to the temperature axis. When the temperature is held constant, the plate current is limited by *space-charge saturation*, sometimes called *voltage saturation*, as the plate voltage is increased. When the plate voltage is held constant, the plate current is limited by *temperature saturation*, as the emitter temperature is increased. The points on the rounded edge of the intersection are limited by both types of saturation.

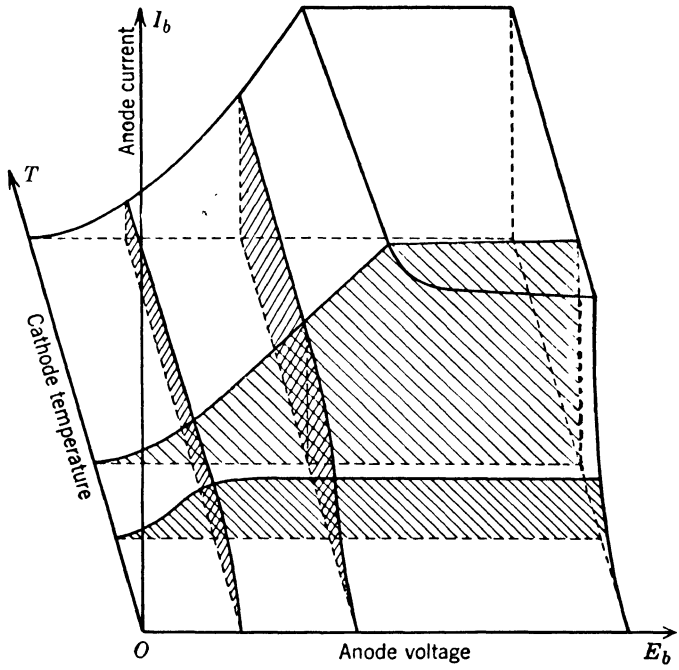


FIG. 17.9. Current-temperature-voltage characteristic surface of a diode.

17.5. Direct-current Resistance of a Diode. In practice the temperature of the emitter is held constant. The operation of the diode plate current will then be a function of only the plate voltage. The operating characteristic will then be of the type shown in Fig. 17.8. This characteristic curve is redrawn in Fig. 17.10 with the curves of the d-c and a-c resistance.

The resistance of a diode to direct current can be determined from Ohm's law in the form

$$r_b = \frac{E_b}{I} \tag{17.4}$$

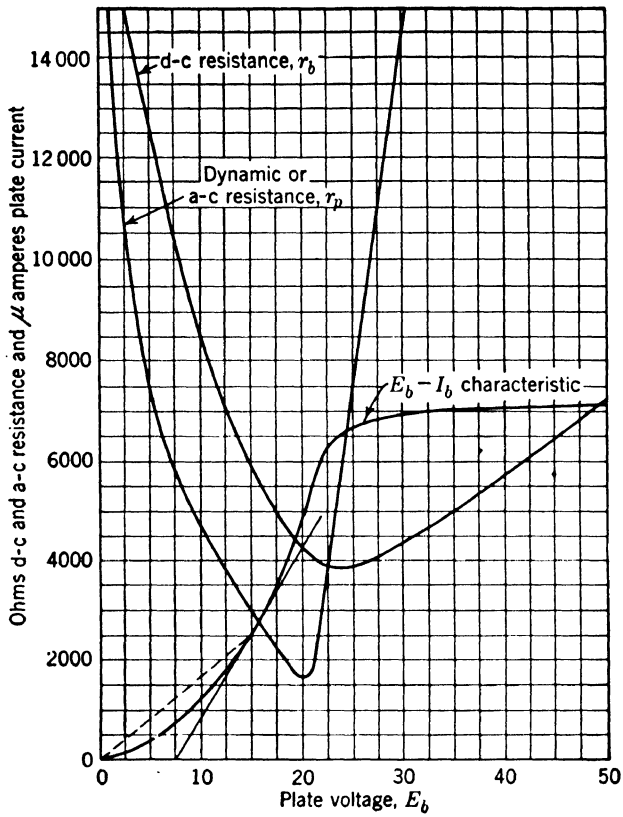


FIG. 17.10. D-c and a-c resistance of an anode.

where E_b = plate voltage, volts
I_b = plate current, amp
r_b = d-c resistance, ohms

Example 17.4. Determine the d-c resistance from the characteristic curve of Fig. 17.10 when the plate voltage is 15 volts.

Solution. For $E_b = 15$ volts, $I_b = 0.0025$ amp and from Eq. (17.4)

$$r_b = \frac{E_b}{I_b} = \frac{15}{0.0025} = 6{,}000 \text{ ohms} \qquad \textit{Ans.}$$

The d-c resistance is the reciprocal of the slope of the dotted line in Fig. 17.10. This is the method used to determine the points on the d-c resistance curve of Fig. 17.10.

17.6. Alternating-current Resistance of a Diode. The a-c resistance of a diode, sometimes called the *dynamic plate resistance,* is the reciprocal of the slope of the e_b-i_b characteristic at the point in question. In general, the a-c resistance is quite different from the d-c resistance, usually having

a smaller value in the operating portion of the curve as shown by the curve in Fig. 17.10.

The a-c or dynamic resistance can be determined from

$$r_p = \frac{\Delta E_b}{\Delta I_b} \tag{17.5}$$

where ΔE_b = a small change in plate voltage at the desired point on the characteristic
ΔI_b = the corresponding change in the plate current
r_p = the a-c or dynamic plate resistance

Example 17.5. Determine the a-c resistance for the characteristic curve of Fig. 17.10 when the plate voltage is $E_b = 15$ volts.

Solution. For $E_b = 15$ volts, $I_b = 0.0025$ amp. Draw a tangent to the curve at this point. This tangent intersects the x-axis at 7.5 volts making a triangle such that $\Delta E_b = 15 - 7.5 = 7.5$ volts and $\Delta I_b = 0.0025 - 0 = 0.0025$ amp. Then

$$r_p = \frac{7.5}{0.0025} = 3{,}000 \text{ ohms} \qquad \textit{Ans.}$$

It will be recalled that this large triangle gives the same results as a small triangle.

Equation (16.5) is sometimes written in differential form as follows

$$r_p = \frac{de_b}{di_b} \tag{17.6}$$

which means that de_b and di_b form the sides of an infinitesimally small triangle with an infinitesimally small section of the curve as the hypotenuse; hence the result is the same as if a line is drawn tangent to the curve and a large triangle formed to determine the slope. Small letters e_b and i_b are used to represent instantaneous values because the operating point is continually shifting.

When other variables are present, such as temperature in this case, the correct way to express the a-c or dynamic plate resistance is

$$r_p = \frac{\partial e_b}{\partial i_b} \tag{17.7}$$

where ∂e_b and ∂i_b, called *partial of* e_b and *partial of* i_b, mean that any other variables, such as temperature in this case, are held constant during this measurement of r_p, the plate resistance.

The plate conductance of a diode is the reciprocal of the dynamic plate resistance, hence

$$g_p = \frac{1}{r_p} = \frac{\partial i_b}{\partial e_b} \tag{17.8}$$

where ∂i_b = the infinitesimal change in plate current corresponding to ∂e_b, the infinitesimal change in plate voltage
r_p = the dynamic plate resistance
g_p = the dynamic plate conductance

The dynamic plate conductance of a diode according to this equation is the slope of the e_b-i_b characteristic curve at the operating point, while the static or d-c plate conductance is the reciprocal of the d-c plate resistance.

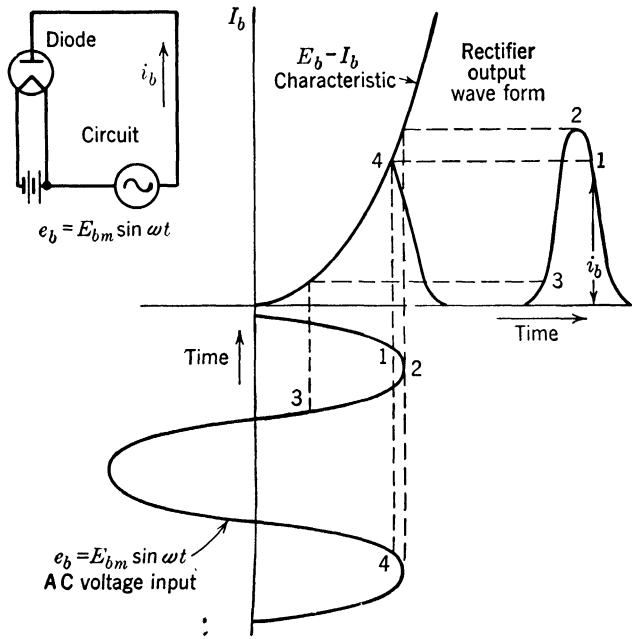


FIG. 17.11. Determination of the a-c plate-current wave form graphically from the diode characteristic when a sinusoidal a-c voltage is applied to the plate.

17.7. Graphical Representation of Input and Output of a Diode. Since the e_b-i_b characteristic curve is *not a straight line*, the resistance does not stay constant, and Ohm's law cannot be applied over the complete range. Such a device is commonly called a *nonlinear impedance*, and may be very useful in certain applications, but in other applications it offers limitations that must be carefully considered.

17.8. Sinusoidal A-c Voltage Input. If a sinusoidal a-c voltage is applied to a diode whose impedance is a pure resistance, the resulting current can be easily obtained graphically by the use of the e_b-i_b characteristic curve as shown in Fig. 10.11. The wave form of the voltage input is plotted along the negative y-axis. For this curve the ordinate is the time axis. The plate current is then drawn with the x-axis as its time axis.

In Fig. 17.11, point 1 on the voltage wave form is projected up to the e_b-i_b characteristic curve and then across to point 1 on the plate-current pulse. This means that when the instantaneous plate voltage is at 1 on the voltage wave, the corresponding instantaneous plate current is at 1 on the current pulse. Similarly points 2, 3, and 4 can be projected from the voltage wave to the current pulse. It will be noted that when the plate voltage is negative (to the left of the y-axis), no current flows. This

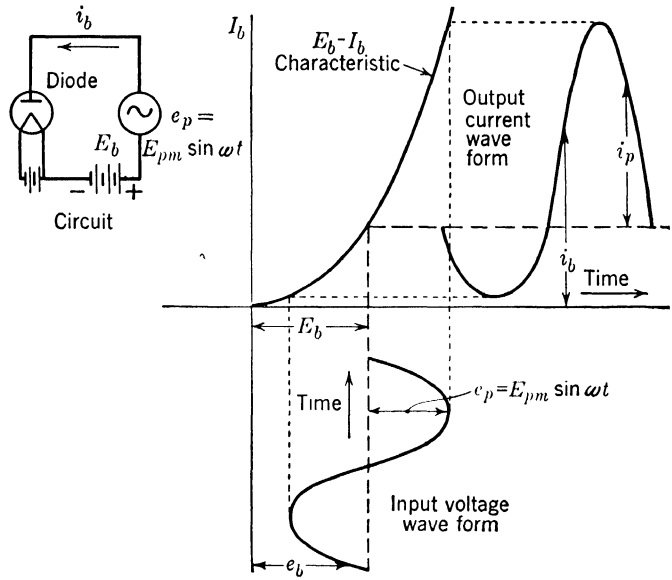


FIG. 17.12. Graphically determining the wave form of the alternating plate current, i_p, from the diode characteristic when an alternating sinusoidal voltage is applied to the plate in series with a positive d-c component of voltage.

merely means that electrons will flow from the cathode to the plate when the plate is positive, but not from the plate to the cathode when the plate is negative. With a-c voltage applied to the plate, current flows only during the time when the plate is positive. Such tubes are therefore useful as rectifiers to convert alternating current to direct current for supplying plate, grid, and screen voltages to other vacuum tubes.

17.9. Sinusoidal A-c Component Plus a D-c Component of the Input Voltage. If the d-c component of the voltage is applied in series with the a-c component of the voltage on the plate of a diode, the axis of the a-c component of the voltage wave will be shifted to the right of the negative y-axis if the d-c component of the voltage is positive and to the left if it is negative. If the voltage is negative, the pulses of plate current will be made smaller until they completely disappear when the d-c component of the voltage is equal to E_{bm}, the maximum value of the a-c

component of the voltage. When the d-c component of the voltage is positive, the pulses of plate current will become larger until the d-c component of the voltage equals E_{bm}. Beyond this point, the plate current will flow at all times. The d-c voltage can then be used to select the most desirable operating point on the E_b-I_b characteristic. Figure 17.12 shows graphically how an a-c component of the voltage superimposed on a d-c component of the voltage causes the plate current to vary. The plate-current curve is obtained by projecting corresponding points from the voltage wave up to the E_b-I_b characteristic and across to the plate-current curve. This is the same procedure that was used above where it was explained in detail.

It will be noted that the plate-current wave form is not symmetrical about its axis; the half cycles above the axis are larger than the half cycles below the axis. This is due to the fact that the E_b-I_b characteristic is not a straight line, giving a nonlinear impedance. This nonlinearity is sometimes used in detectors of radio circuits.

Summary

A diode tube consists of an electron emitter, or cathode, and a collecting plate, or anode. Current will flow through a diode only when the anode is at a positive potential with respect to the cathode.

The total electron emission current from the cathode of a diode tube is given by Richardson's equation

$$I_S = AT^2\epsilon^{-b_0/T}$$

where I_S = current, amp/sq cm
A = 60.2, a constant for all pure metals
= 3 for thoriated tungsten
= 10^{-2} for barium-strontium oxides
T = absolute temperature, °K
ϵ = 2.71828 (the Napierian base of natural logarithms)
$b_0 = \frac{\phi_0}{K}$ = a constant depending upon the work necessary to move an electron from the inside to the outside of the emitter
ϕ_0 = work function, electron-volts
K = 8.63×10^{-5} = Boltzmann's constant, volts/°K or °C

When the anode is not sufficiently positive to collect all the electrons emitted by the cathode, the excess electrons form a cloud around the cathode, which is known as a *space charge.*

The plate current for a diode in which the cathode and anode are parallel plates of infinite extent, and in which a space charge exists, is given by Child's law

$$I_b = \frac{A(2.34 \times 10^{-6})E_b^{3/2}}{S^2}$$

where I_b = plate or anode current, amp
A = area of the plate, sq cm
E_b = plate or anode voltage, volts
S = spacing between the cathode and anode, cm

The plate current for a diode in which the anode and cathode consist of infinitely long coaxial cylinders, and in which space charge exists, is given by Langmuir's law

$$I_p = \frac{l(14.68 \times 10^{-6})E_b^{3/2}}{r\beta^2}$$

where I_b = plate or anode current, amp
l = length of the cylinders, cm
E_p = plate voltage, volts
r = radius of the plate, cm
β^2 = 1, which gives an accuracy within a few per cent when the ratio of the diameter of the anode to the diameter of the cathode is greater than 100

An electron tube that is operated under conditions such that the anode collects all the electrons produced by the emitter is said to be *temperature limited.*

An electron tube operated with the anode at a potential lower than that necessary to collect all the electrons emitted by the cathode is said to be *space charge limited.*

The d-c plate resistance of a diode is given by

$$r_b = \frac{E_b}{I_b}$$

where E_b = plate voltage, volts
I_b = plate current, amp
r_b = d-c resistance, ohms

The a-c or dynamic plate resistance of a diode is given by

$$r_p = \frac{\Delta E_b}{\Delta I_b}$$

where ΔE_b = a small change in plate voltage at the desired point on the characteristic, volts
ΔI_b = the corresponding change in the plate current, amp
r_p = the a-c or dynamic plate resistance, ohms

This equation can be written in differential form as

$$r_p = \frac{de_b}{di_b}$$

When other variables such as temperature are present, the a-c plate resistance is written as

$$r_p = \frac{\partial e_b}{\partial i_b}$$

where ∂e_b and ∂i_b, called *partial of* e_b and *partial of* i_b, mean that any other variables, such as temperature in this case, are held constant during this measurement of r_p, the plate resistance.

The plate conductance of a diode is the reciprocal of the dynamic plate resistance.

Exercises

17.1. What is unilateral conductivity? Name several devices that have this property.

17.2. A thoriated-tungsten filament has an area of 0.175 sq cm. Determine the emission current if the temperature is 1600°K and the work function is 2.63 electron-volts.

17.3. How is the space charge distributed when the anode is connected to the cathode?

17.4. How does the plate current depend upon the plate voltage? Who was the first to develop this relation? What are the assumptions?

17.5. Determine the plate current when the plate is a cylinder 2 cm long and 1.5 cm in diameter and the plate voltage is 200 volts. Assume the cathode to be a very small cylinder.

17.6. How does the crest or maximum point of space charge shift when the plate voltage is increased?

17.7. Why is it convenient to view a diode's characteristic in three dimensions?

17.8. Why is the d-c resistance of a diode different from the a-c resistance? Explain.

17.9. Two points on an $E_b - I_b$ characteristic are

$$E_{b1} = 200 \text{ volts} \qquad E_{b2} = 220 \text{ volts}$$
$$I_{b1} = 65 \text{ ma}, \qquad I_{b2} = 68 \text{ ma}$$

Determine the d-c resistance for both points.

17.10. Determine the a-c resistance for the points of Exercise 17.9, assuming a straight line between the two points.

CHAPTER 18

THE TRIODE TUBE

Thus far we have studied the electron, methods of producing a supply of electrons, and the effect of introducing a plate to collect the electrons. This chapter will be devoted to a study of the effects brought about by introducing a third element known as a *grid* to control the streams of electrons between the cathode and anode. Such a three-element tube is known as a *triode*.

18.1. Historical Sketch. Before 1907 the only vacuum tubes in existence were two-electrode tubes, or diodes, which could be used only as power rectifiers or as the so-called *Fleming valve detectors*. In 1907, Lee De Forest made the remarkable discovery that by inserting a third electrode between the filament and plate, consisting of small wires forming a mesh or grid, the flow of electrons between the filament and plate could be readily controlled. This greatly extended the use of vacuum tubes for applications such as a-f and r-f amplifiers, detectors, and oscillators.

Van der Bijl was one of the first to give the three-electrode tube a thorough theoretical mathematical treatment. Most subsequent treatments have been based upon this work.

18.2. Grid Control of Electron Stream. The primary purpose of the grid is to control the flow of plate current by inserting a charge between the cathode and anode. This should be done with a minimum mechanical obstruction to electron flow and without absorbing any electrons. The first requirement is fulfilled by making the grid with as much free space as possible, and the second requirement is met by operating the grid at a negative potential with respect to the cathode so it will not collect electrons.

The potential of the grid determines the resultant electrostatic field between the cathode and the anode. Figure 18.1 shows the field due to various grid potentials. The complete electrostatic field is secured by superimposing the space-charge field of the electrons on the fields caused by the electrodes.

Figure 18.1*a* shows how a positive grid assists the anode in creating a field that will move large numbers of electrons from the cathode to the anode. If the grid is made too positive, the electrons will flow to the grid and actually reduce the anode current. Figure 18.1*b* shows that the anode will attract some electrons even if the grid has no attraction,

being at the same potential as the cathode. Figure 18.1*c* corresponds to the case of a diode; the free grid collects just enough electrons to establish the potential of the space charge, hence the effect of the grid on the field

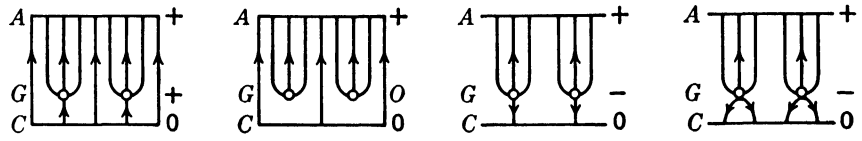
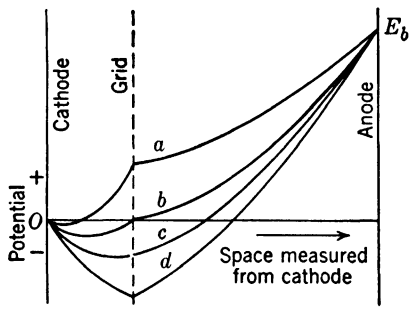


FIG. 18.1. Electrostatic field in a triode.

is nil. This condition is very unstable and should not be used in practice. In Fig. 18.1*d* the grid is made so negative that it forces electrons back toward the cathode and reduces the number reaching the anode.

18.3. Potential Distribution in a Triode. When the electrostatic fields due to the electrodes and space charge are superimposed, the potential distribution can be represented as shown in Fig. 18.2.



a = positive grid *c* = free grid
b = zero grid *d* = negative grid

FIG. 18.2. Potential distribution in a triode.

When the grid is positive, the negative space charge (represented by the portion of the potential line below the zero-potential axis) is very small and very close to the cathode. This means that most of the electrons are drawn away from the cathode, some going to the grid and the rest to the anode. With the grid at cathode potential, there is a negative space charge between the cathode and grid. The electrons that pass through the grid are attracted to the plate. If the grid is free, it will assume a negative charge, resulting in the same curve as that of the diode, which is shown in Fig. 17.7. This is a very unstable condition of operation, as the grid may collect an excessive negative charge and block the plate current. By making the grid more negative, the negative space-charge effect is increased to reduce the number of electrons passing the grid and going to the anode.

If it were not for the electron space charge, all the potential lines of Fig. 17.2 would be straight lines. The negative charge of the electrons between the electrodes makes all these curves bend downward.

From this analysis, the effect of the grid can be briefly stated as follows:

1. Positive grid: Decreases the space charge. Increases anode current. Increases grid current. Decreases anode-cathode resistance.

2. Free grid: No effect—same as a diode. (Very unstable in practice, may block plate current.)

3. Negative grid: Increases the space charge. Decreases anode current. Increases anode-cathode resistance.

The anode-cathode resistance, usually called the *plate resistance*, is determined in the same way as illustrated in the last chapter for a diode. The grid voltage is held constant for this measurement.

18.4. Static Characteristic Curves of a Triode. There are three independent variables in a triode, namely, the plate, grid, and filament voltages. In most applications, the filament voltage is held constant at a value that will give the required supply of electrons from the cathode.

There are also three dependent variables: plate, grid, and filament currents. Since the filament voltage is usually held constant, the filament current will also be a constant. This leaves four important parameters, two of which are independent (E_c, E_b), and two of which are dependent (I_b, I_c). Normally, the triode is operated to reduce the grid-current effects, if not to eliminate them entirely, by keeping the grid negative.

The most important static characteristics are the *transfer*, *plate*, and *constant-current characteristics*, indicated by the solid curves in Figs. 18.3, 18.5, and 18.6 respectively. The transfer characteristics, sometimes called the E_c-I_b (*grid-voltage–plate-current*) *curves*, are secured by measuring the plate current for various values of grid voltage while the plate voltage is held constant. A family of such curves for various values of plate voltage is shown by the solid curves in Fig. 18.3. The plate characteristics, sometimes referred to as the E_b-I_b (*plate-voltage–plate-current*) *curves*, are secured by measuring the plate current for various values of plate voltage while the grid voltage is held constant. A family of such curves for various values of grid voltages is shown by the solid curves of Fig. 18.5. The constant-current characteristics, sometimes referred to as the E_b-E_c (*plate-voltage–grid-voltage*) *curves*, can be secured by measuring the grid voltage for various values of plate voltage when the plate current is held constant. A family of such curves for various values of plate current is shown in Fig. 18.6. It will be noted that these important families of static characteristic curves involve two independent variables and one dependent variable (E_b, E_c, I_b). Knowing the value of any two of these variables, it is possible to determine the other one. In other words, the three families of solid curves shown in Figs. 18.3, 18.5, and 18.6 are plots of the same information in three different forms to increase its usefulness.

These static characteristic curves give the *true instantaneous values* of the relation between the three variable, E_b, E_c, and I_b, except for very

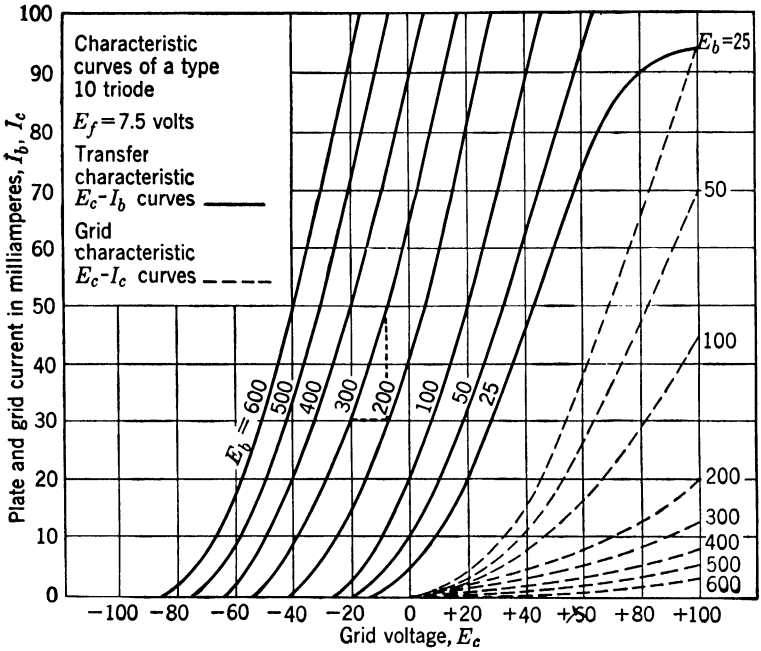


FIG. 18.3. Transfer characteristic curves of a triode.

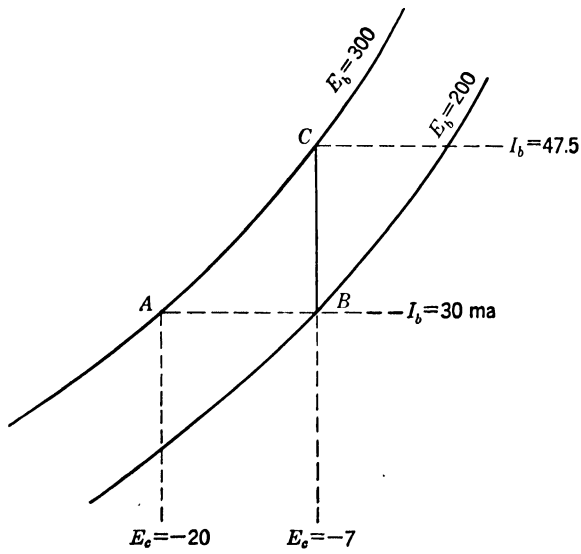


FIG. 18.4. Expanded portion of characteristic curves from Fig. 18.3.

high frequencies above 100 Mc, where the transit time of the electrons in the tube must be given due consideration. This means that *graphical methods* based on the static characteristics readily give exact solutions to many practical problems. The transfer characteristics, or E_b-I_c curves, for example, are valuable for determining the shape of the plate-current or I_b wave form resulting from a given grid-voltage on E_c wave form when the plate voltage E_b is a constant. The most useful characteristic

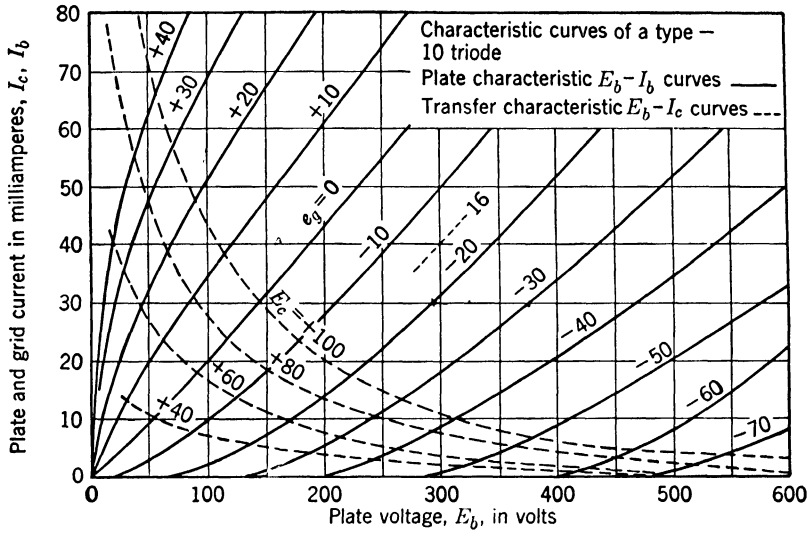


FIG. 18.5. Plate characteristic curves of a triode.

is the plate characteristic or E_b-I_b curves. These are the most commonly published static characteristic curves, because even with a reactive load, the locus of E_b-I_b values gives a straight line, and because certain areas of this diagram are proportional to the power supplied and expended in the plate circuit. The constant-current characteristics, or E_b-E_c curves, have been found to be very useful in simplifying the analysis of class C amplifiers.

If the grid is to be operated with positive grid voltages, it is important to have the grid, grid-transfer, and constant-grid-current characteristics, as shown by the dashed curves in Figs. 18.3, 18.5, and 18.6, respectively. The grid characteristics, or E_c-I_c curves, dashed in Fig. 18.3, show how the grid current i_g varies as a function of the grid voltage E_c, when the plate voltage E_b is held constant. The grid-transfer characteristics, or E_b-I_c curves, dashed in Fig. 18.5, show how the grid current I_c varies as a function of the plate voltage E_b, when the grid voltage E_c is held constant. Finally, the constant-grid-current characteristics, or E_b-E_c curves,

dashed in Fig. 18.6, show how the grid voltage E_c varies as a function of the plate voltage E_b, when the grid current I_c is held constant. It should be noted that these families of dashed curves involve two independent variables and one dependent variable (E_c, E_b, E_c). The information has been presented in three different forms to increase its usefulness. Just as with the other characteristic curves, graphical methods can be used with these curves to arrive readily at exact solutions. Mathematical solutions are not easy because of the complexity of the equations.

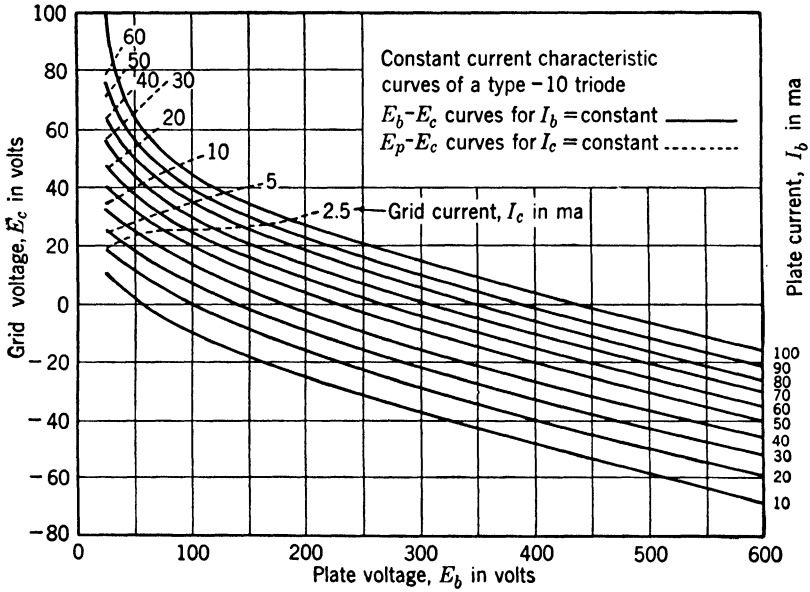


FIG. 18.6. Constant current characteristic curves of a triode.

18.5. Triode Constants. The solution of vacuum-tube problems is in most cases simplified by the use of three derived factors that depend upon the physical construction and upon the electrode voltages and currents. These three factors, termed the *amplification factor* μ (Greek letter *mu*), *the dynamic plate resistance* r_p, and the *mutual conductance* g_m, make possible the solution of many problems without resort to the static characteristics.

18.6. Amplification Factor μ. Theoretically the amplification factor can be made to depend entirely upon the geometry of the tube, but actually it varies somewhat with electrode voltages. Hence it can be defined as *the ratio of the change in plate voltage,* dE_b, *to the change in grid voltage,* dE_c, *that causes the change in* E_b, *when all other parameters, such as plate current* i_p, *are held constant.* This definition, incidentally, is valid

for other tubes than the triode. Mathematically, this definition can be written

$$\mu = -\frac{dE_b}{dE_c} \qquad (i_b = \text{constant}) \qquad 18.1)$$

where μ = amplification factor (always positive)
dE_b = incremental change in plate voltage, volts
dE_c = incremental change in grid voltage, volts

Another way of writing this equation, which has the same meaning, is by means of partial differentials; thus

$$\mu = -\frac{\partial E_b}{\partial E_c} \qquad (18.1a)$$

Graphically, the amplification factor μ can be determined from Fig. 18.3 by holding the plate current constant and measuring the change in plate voltage corresponding to a given change in grid voltage. Similarly, this value can be determined from Figs. 18.5 and 18.6. The amplification factor is very useful in calculating the voltage gain that can be expected from a given vacuum tube. It should be noted that $\partial E_b/\partial E_c$ must be negative since μ is always positive.

Example 18.1. Graphically determine the amplification factor μ for the type 10 triode by means of Figs. 18.3, 18.5, and 18.6.

Solution. Referring to Fig. 18.3 for $i_p = 20$ ma, the approximate values are

$$dE_b \doteq E_{b1} - E_{b2} = 300 - 200 = 100 \text{ volts}$$

and the corresponding change in grid voltage is

$$dE_c \doteq E_{c1} - E_{c2} = -28 - (-15) = -13 \text{ volts}$$

Therefore, by Eq. (18.1) the amplification factor μ is

$$\mu = -\frac{dE_b}{dE_c} \doteq -\frac{100}{-13} = 7.69 \qquad \textit{Ans.}$$

Referring to Fig. 18.5 for $i_p = 20$ ma, let us approximate by writing

$$dE_c \doteq E_{c1} - E_{c2} = -20 - (-30) = +10 \text{ volts}$$

and the corresponding change in plate voltage is

$$dE_b \doteq E_{b1} - E_{b2} = 237 - 314 = -77 \text{ volts}$$

Then by Eq. (18.1) the amplification factor μ is

$$\mu = -\frac{dE_b}{dE_c} \doteq -\frac{-77}{10} = 7.7 \qquad \textit{Ans.}$$

Referring to Fig. 18.6 for $i_p = 20$ ma,

$$dE_b \doteq E_{b1} - E_{b2} = 200 - 300 = -100 \text{ volts}$$

Then the corresponding change in grid voltage is

$$dE_c \doteq -15 - (-28) = 13 \text{ volts}$$

and

$$\mu = -\frac{dE_b}{dE_c} \doteq -\frac{-100}{13} = 7.69 \text{ volts} \qquad \textit{Ans.}$$

Or if we had taken

$$dE_c \doteq E_{c1} - E_{c2} = -20 - (-30) = +10 \text{ volts}$$

the corresponding change in plate voltage would be

$$dE_b \doteq E_{b1} - E_{b2} = 237 - 314 = -77 \text{ volts}$$

and

$$\mu = -\frac{dE_b}{dE_c} \doteq -\frac{-77}{10} = 7.7 \qquad \textit{Check}$$

18.7. Dynamic Plate Resistance r_p. Like the amplification factor, the dynamic plate resistance, commonly referred to simply as the *plate resistance*, depends upon the geometry of the tube. The plate resistance to alternating current may be quite different from the d-c plate resistance, just as in the case of a diode, which was explained in the last chapter. The dynamic plate resistance refers to incremental changes in the plate voltage and current and can be defined as *the ratio of change in plate voltage, dE_b, to the change in plate current, dI_b, that causes the change in E_b, when all other parameters, such as grid voltages, are held constant.* This definition is, of course, valid for all vacuum tubes. Mathematically, this definition can be written

$$r_p = \frac{dE_b}{dI_b} \qquad (e_c = \text{constant}) \qquad (18.2)$$

where r_p = dynamic plate resistance, ohms
dE_b = incremental change in plate voltage, volts
dI_b = incremental plate current, amp

The plate resistance can be determined graphically from Figs. 18.3, 18.5, and 18.6 by holding the grid voltage constant for the determination.

The plate resistance can also be expressed by partial differentials; thus,

$$r_b = \frac{\partial E_b}{\partial I_b} \qquad (18.2a)$$

Example 18.2. Determine the plate resistance of a type 10 triode when the grid voltage $E_c = -20$ volts.

Solution. From Fig. 18.3 we have a change of

$$dI_b \doteq I_{b1} - I_{b2} = 31.5 - 14 = 17.5 \text{ ma}$$

when the plate voltage varies by

$$dE_b \doteq E_{b1} - E_{b2} = 300 - 200 = 100 \text{ volts}$$

Then by Eq. (18.2) we can write

$$r_p = \frac{dE_b}{dI_b} \doteq \frac{100}{0.0175} = 5{,}710 \text{ ohms} \qquad \textit{Ans.}$$

18.8. Mutual Conductance g_m. The mutual conductance of a vacuum tube is the ratio of a small change in plate current, dI_b, to the small change in grid voltage, dE_c, that causes the change in I_b, when the other param-

eters, such as the plate voltage dE_b, are held constant. Mathematically this definition can be written

$$g_m = \frac{dI_b}{dE_c} \qquad (e_b = \text{constant}) \qquad (18.3)$$

where g_m = mutual conductance, mhos
dI_b = incremental change in plate current, amp
dE_c = incremental change in grid voltage, volts

The mutual conductance expressed in terms of partial differentials becomes

$$g_m = \frac{\partial I_b}{\partial E_c} \qquad (18.3a)$$

Now let us investigate the relationship between μ, r_p, and g_m. First, let us permit a small positive change in grid voltage with the plate voltage remaining constant; thus from Eq. (18.3),

$$dI_b = g_m \, dE_c$$

which means that the plate current changes a small positive amount. This fact can be verified by referring to Fig. 18.3. Since a small positive change in E_c causes a small positive change in I_b then g_m must be a positive quantity.

Example 18.3. Determine the mutual conductance of a Type 10 triode from Fig. 18.3 when the plate voltage is held at 300 volts.

Solution. Referring to Fig. 18.3,

$$dI_b \doteq I_{b1} - I_{b2} = 30 - (20) = 10 \text{ ma}$$
when $$dE_c \doteq E_{c1} - E_{c2} = -21 - (-28.5) = 7.5 \text{ volts}$$

Then, by Eq. (18.3)

$$g_m = \frac{dI_b}{dE_c} \doteq \frac{0.010}{7.5} = 1{,}333 \ \mu\text{mho} \qquad \textit{Ans.}$$

Using values of μ and r_p determined above, Eq. (18.6) gives

$$g_m = \frac{\mu}{r_p} = \frac{7.69}{5{,}710} = 1{,}346 \ \mu\text{mho} \qquad \textit{Check}$$

For approximate graphical solutions, this is a reasonably accurate check.

Referring to Fig. 18.3 consider Eq. (18.1). For a constant plate current of $I_b = 30$ ma let the plate voltage change from $E_b = 300$ volts to $E_b = 200$ volts as shown in Fig. 18.4. This decrease in plate voltage for constant plate current results in an increase of grid voltage from $E_c = -20$ volts to $e_g = -7.0$ volts. If the incremental quantities dE_b and dE_c of Eq. (18.1) are made large for illustration purposes this equation can be used to determine the amplification factor from the preceding data, thus,

$$\mu = -\frac{300 - 200}{-20 - (-7.0)} = 7.7$$

Next, hold the grid voltage constant at $E_c = -7.0$ and increase the plate voltage from $E_b = 200$ volts to $E_b = 300$ volts. Referring to Fig. 18.4 it is seen that the plate current increases from $I_b = 30$ ma to $I_b = 47.5$ ma. Now, if the incremental quantities of Eq. (18.2) are made large and the above values substituted, the plate resistance is determined to be

$$r_p = \frac{200 - 300}{0.030 - 0.0475} = 5{,}710 \text{ ohms}$$

Finally, hold the plate voltage constant at $E_b = 300$ volts and decrease the plate current from $I_b = 47.5$ ma to $I_b = 30$ ma. In a similar fashion by Eq. (18.3) the mutual conductance is computed to be

$$g_m = \frac{0.0475 - 0.03}{-7.5 - (-20)} = 1{,}400 \ \mu\text{mho}$$

With the triangle of Fig. 18.4 in mind, multiply r_p by g_m as given in Eq. (18.2) and Eq. (18.3). This multiplication gives,

$$r_p g_m = \frac{dE_b}{dI_b} \times \frac{dI_b}{dE_c} = \left(\frac{dE_b}{dE_c}\right) \qquad (i_b = \text{constant})$$

In the first term on the right hand side of this equation the current increases a small amount and in the second term it returns to its original value, therefore, for constant plate current these terms cancel leaving $-dE_b/dE_c$ which, by Eq. (18.1), is the definition of μ. The primary purpose of this discussion is to illustrate the necessity of the negative sign in Eq. (18.1) and to show that the correct solution of Eqs. (18.1), (18.2) and (18.3) can be expressed in the following very useful forms

$$\mu = r_p g_m \tag{18.4}$$

$$r_p = \frac{\mu}{g_m} \tag{18.5}$$

$$g_m = \frac{\mu}{R_p} \tag{18.6}$$

These equations show that if any two of the constants are known, the other can be readily calculated.

In terms of partial differential equations the relationship between μ, r_p and g_m can be determined rigorously by letting the grid and plate voltages change small amounts and write down the corresponding change in plate current, thus

$$I_b = \frac{\partial I_b}{\partial E_c} dE_c + \frac{\partial I_b}{\partial E_b} dE_b$$

Since $g_m = \dfrac{\partial I_b}{\partial E_c}$ and $r_p = \dfrac{\partial E_b}{\partial I_b}$, we can rewrite the preceding equation

$$I_b = g_m\, dE_c + \frac{1}{r_p}\, dE_b$$

If the change in plate current is held to zero this equation can be solved for

$$g_m r_p = -\left(\frac{dE_b}{dE_c}\right)(i_b = \text{constant}) = \mu$$

This relationship is exact for all tubes and any combination of electrode potentials.

18.9. Simplified Equivalent Circuit of a Triode. The triode constants can be used to an advantage in reducing a triode circuit to a simplified equivalent circuit. To simplify the analysis, let the grid remain negative

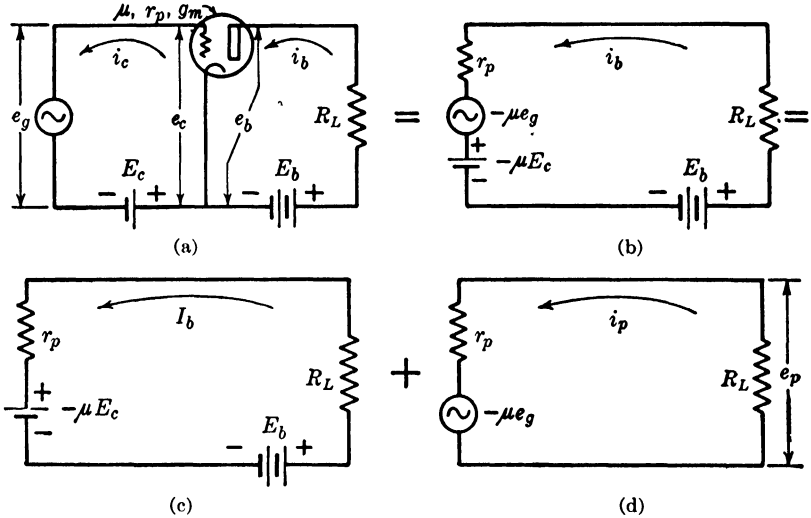


FIG. 18.7. Simplified equivalent circuits of a triode with a resistance load.

at all times; thus it will consume no power but will be able to control considerable power in the plate circuit. Furthermore, if the triode constants are to be used, it will be necessary to operate on the straight-line portion of the transfer characteristics shown in Fig. 18.3.

One of the fundamental theorems known as the *equivalent-circuit theorem* can be used to replace the varying plate resistance, which is difficult to treat mathematically, by a constant resistance and a varying emf. The theorem, applied to this problem, says that *small changes in plate current I_b can be calculated by assuming that the voltage $-\mu E_c$ acts in series with a constant plate resistance r_p.*

By using the *superposition theorem*, the total instantaneous plate current i_b, as given in Fig. 18.7*b*, is equal to the sum of the d-c component of plate current, I_b, as given in Fig. 18.7*c*, plus the a-c component of plate

current, i_p, as given in Fig. 18.7*d*; thus

$$i_b = I_b + i_p \tag{18.7}$$

where i_b = total instantaneous plate current
I_b = d-c component of plate current
i_p = instantaneous a-c component of plate current

This current relationship is also illustrated in Fig. 18.8.

The total instantaneous grid voltage by reference to Fig. 18.7*a* and Fig. 18.8 is

$$\begin{aligned} e_c &= E_c + e_g \\ &= E_c + E_{gm} \sin \omega t \end{aligned} \tag{18.8}$$

where e_c = total instantaneous grid voltage
E_c = d-c component of grid bias voltage
$e_g = E_{gm} \sin \omega t$ (18.9)
= instantaneous a-c component of grid voltage
E_{gm} = maximum value of a-c component of grid voltage
$\omega = 2\pi f$ = angular velocity, radians/sec
t = time, sec

Applying Kirchhoff's voltage law to the equivalent d-c plate circuit of Fig. 18.7*c* gives,

$$I_b r_p - \mu E_c - E_b + I_b R_L = 0 \tag{18.10}$$

Solving this equation for the d-c component of plate current,

$$I_b = \frac{+\mu E_c + E_b}{r_p + R_L} \tag{18.11}$$

It should be noted that E_c is normally a negative voltage, otherwise it would be expressed with a plus sign in Eq. (18.10).

Now applying Kirchhoff's voltage law to the equivalent a-c plate circuit of Fig. 18.7*d* gives,

$$i_p r_p + \mu e_g + i_p R_L = 0 \tag{18.12}$$

Solving this equation for

$$i_p = \frac{\mu e_g}{r_p + R_L} \tag{18.13}$$

An inspection of this equation and Fig. 18.8 reveals that if the a-c component of the grid voltage e_g increases (becomes less negative in the grid circuit) the a-c component of the plate current increases.

By substituting Eq. (18.11) and Eq. (18.13) in Eq. (18.7) the total instantaneous plate current is

$$i_b = \frac{\mu e_c + E_b}{r_p} = \frac{\mu E_c + \mu e_g + E_b}{r_p + R_L} \tag{18.14}$$

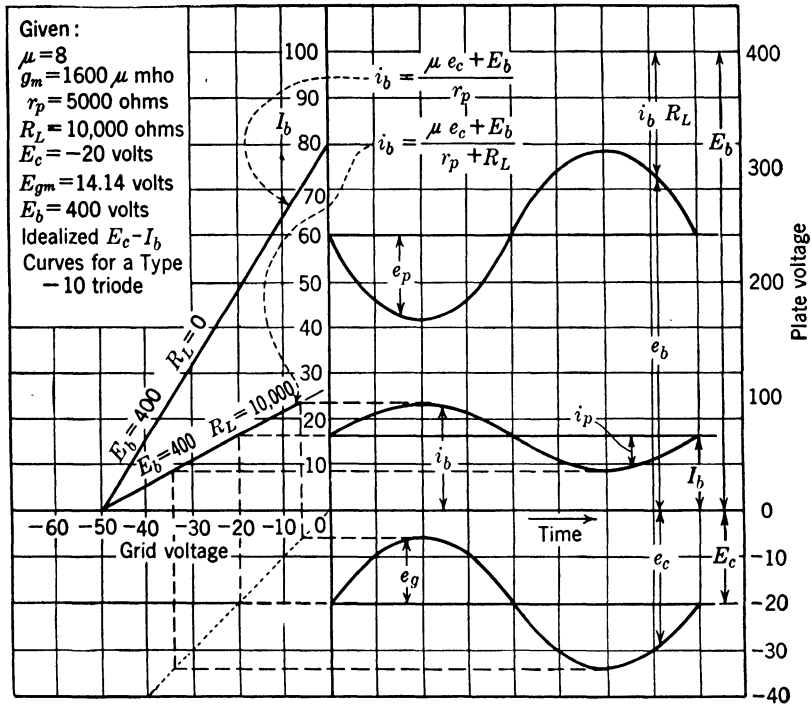
The total instantaneous plate voltage by reference to Fig. 18.7*a* and Fig. 18.8 is

$$e_b = E_b - i_b R_L \tag{18.15}$$

Substituting i_b in Eq. (18.15) with the value given in Eq. (18.14) results in,

$$e_b = \frac{E_b r_p - \mu E_c R_L}{r_p + R_L} - \frac{\mu R_L}{r_p + R_L} e_g \tag{18.16}$$

$$= \frac{E_b r_p - \mu E_c R_L}{r_p + R_L} - \frac{\mu R_L E_{gm}}{r_p + R_L} \sin \omega t \tag{18.17}$$



Solution:

$$e_g = E_c + E_{gm} \sin \omega t = -20 + 14.14 \sin \omega t$$

$$e_p = \frac{E_b R_p - \mu E_c R_L}{R_p + R_L} - \frac{\mu E_{gm} R_L}{R_p + R_L} \sin \omega t$$

$$= 240 - 75.4 \sin \omega t$$

$$-i_p = \frac{\mu E_c + E_b}{R_p + R_L} + \frac{\mu E_{gm}}{R_p + R_L} \sin \omega t = 16 + 7.54 \sin \omega t$$

$$I_p = \frac{-\mu E_g}{R_p + R_L} = -5.33 \text{ ma}$$

$$A = \frac{-\mu R_L}{R_p + R_L} = -5.33 = \text{useful amplification}$$

FIG. 18.8. Current and voltage relations in a triode operating into a resistance load.

The instantaneous a-c component of the output voltage is,

$$e_p = -i_p R_L \tag{18.18}$$

Substituting for i_p from Eq. (18.13) and for e_g from Eq. (18.9) gives

$$e_p = \frac{-\mu E_{gm} R_L}{r_p + R_L} \sin \omega t \tag{18.19}$$

the instantaneous a-c component of the output voltage expressed in terms of the input voltage and pertinent constants of the vacuum tube circuit.

18.10. Important Vacuum Tube Equations. Perhaps the most important equation of vacuum tube engineering is the a-c component of the plate current written in terms of effective values. Eq. (18.13) becomes,

$$I_p = \frac{\mu E_g}{r_p + R_L} \tag{18.20}$$

where I_p = effective value of a-c component of the plate current
E_g = effective value of a-c component of input grid voltage

By the application of Ohm's law, the output voltage across the load is,

$$E_L = -I_p R_L = \frac{-\mu E_g R_L}{r_p + R_L} \tag{18.21}$$
$$= \text{effective value of a-c component of output load voltage}$$

Perhaps the most generally useful design equation of vacuum tube engineering can be secured by rewriting Eq. (18.21) thus,

$$A = \frac{E_L}{E_g} = \frac{-\mu R_L}{r_p + R_L} \tag{18.22}$$
$$= \text{useful amplification}$$

The negative sign in this equation, besides being mathematically correct, is a helpful reminder of the 180° phase shift in the triode.

Example 18.4. A Type 10 triode circuit has the following values given:

$\mu = 8$,	$E_c = -20$ volts
$g_m = 1{,}600$ μmho,	$E_{gm} = 14.14$ volts
$R_p = 5{,}000$ ohms,	$E_b = 400$ volts
$R_L = 10{,}000$ ohms	

Draw a diagram illustrating the idealized $E_c - I_b$ curves with a graphical representation of the instantaneous grid and plate voltages and instantaneous plate current. Write the mathematical equations for the instantaneous grid and plate voltages and instantaneous plate current. Determine the value of effective plate current and useful amplification.

Solution. The diagram and solution are given in Fig. 18.8. *Ans.*

Example 18.5. Determine the power consumed by the load resistance R_L of Example 18.4.

Solution. By Joule's law the a-c power is

$$P_{ac} = I_p^2 R_L = (-0.00533)^2(10{,}000) \qquad \textit{Ans.}$$
$$= 0.284 \text{ watt} \qquad \textit{Ans.}$$

and the d-c power is

$$P_{dc} = I_b^2 R_L = (-0.016)^2(10{,}000)$$
$$= 2.56 \text{ watts} \qquad \textit{Ans.}$$

and the total power consumed is

$$P_t = P_{ac} + P_{dc} = 0.284 + 2.56$$
$$= 2.844 \text{ watts} \qquad \textit{Ans.}$$

Summary

The advent of the triode with its control grid makes this type of vacuum tube a control impedance that has revolutionized the radio and communication industry. Its usefulness comes from the fact that relatively large amounts of plate current and power can be controlled with very small variations of grid voltage and practically no expenditure of power in the control circuit.

From the static characteristic curves the true instantaneous values can be determined by graphical methods to give exact solutions, except for frequencies so high that the electron transit time has to be considered.

For most practical calculations the triode constants can be used to simplify the mathematics materially. By utilizing the equivalent-circuit theorem and the superposition theorem, equations of the instantaneous grid and electron plate voltages and plate current can be expressed in the form of an equation.

Also, the effective plate current as a function of the effective grid voltage and the useful amplification are readily expressed in equation form.

As shown in Fig. 18.8, it should be emphasized that

1. e_c and i_b rise and fall together, hence are in phase.
2. e_b falls when i_b rises, hence e_b and i_b are 180° out of phase.
3. e_b falls when e_c rises, hence e_b and e_c are 180° out of phase when the triode is operated into a resistance load. The negative signs take care mathematically of the phase shifts, but are often omitted for the sake of simplicity. The term i_b in this chapter expresses the total current as it flows from anode to cathode and thence through the plate supply and load resistance back to the anode. Therefore, i_b is the conventional direction of current flow.

A triode tube contains, in addition to an anode and a cathode, a third element called *the control grid.* This control grid is located between the cathode and anode, and can be used to control the plate current without appreciable power being consumed in the grid circuit.

The amplification factor μ is given by

$$\mu = -\frac{dE_b}{dE_c} \qquad (i_b = \text{constant})$$

where μ = amplification factor
dE_b = incremental change in plate voltage, volts
dE_c = incremental change in grid voltage, volts

The dynamic plate resistance r_p is given by

$$r_p = \frac{dE_b}{dI_b} \qquad (e_c = \text{constant})$$

where r_p = dynamic plate resistance, ohms
dE_b = incremental change in plate voltage, volts
dI_b = incremental electron plate current, amp

The mutual conductance g_m is given by

$$g_m = \frac{dI_b}{dE_c} \qquad (e_b = \text{constant})$$

where g_m = mutual conductance, mhos
dI_b = incremental change in electron plate current, amp
dE_c = incremental change in grid voltage, volts

Exercises

18.1. Define a triode. How does the grid voltage affect the plate resistance?

18.2. Referring to the $E_c - I_b$ curves for $E_b = 25$ volts in Fig. 18.3, state the reason for the curvature at the lower and upper ends of the curve.

18.3. Of what value are the characteristic curves involving grid current? How does the grid resistance vary with grid voltage?

18.4. When are the static characteristics in error? What can be used to approximate the characteristics involving plate current?

18.5. Graphically determine, from Fig. 18.3, the amplification factor μ between $E_b = 25$ and 50 volts for $i_b = 20$ ma. Determine the plate resistance for these plate voltages when $e_c = +10$ volts. From these values determine the mutual conductance.

18.6. A power-amplifier triode (2A3) connected as shown in Fig. 18.7*a* has the following facts given:

$\mu = 4.2$, $E_c = -45$ volts
$r_p = 800$ ohms, $E_b = 400$ volts
$R_L = 2{,}500$ ohms, $E_g = 30$ volts

Determine the mutual conductance and give the mathematical equations for the idealized $E_c - I_b$ curves with and without the load resistance.

18.7. Give the mathematical equations for the instantaneous grid and plate voltages for Exercise 18.6.

18.8. Write the mathematical equation for the total instantaneous plate current for Exercise 18.6.

18.9. Determine the effective plate current and useful amplification for the power-amplifier triode of Exercise 18.6.

18.10. Give the a-c, d-c, and the total power consumed in the plate and load resistance of the tube circuit of Exercise 18.6. If the tube could be operated into a perfect transformer of the same impedance, how much could the plate voltage be lowered to give the same results and what would be the saving in plate-circuit power?

CHAPTER 19

MULTIELEMENT TUBES

The addition of more than one grid to a vacuum tube gives many desirable characteristics. Since the fundamental purpose of the vacuum tube is to generate, control, and collect a stream of electrons, the action should be *entirely unilateral.* Owing to electrode capacity effects, the triode, unless externally neutralized, has some bilateral properties. The *tetrode,* which has a screen grid between the control grid and plate, was introduced to shield the control grid from the plate electrostatically. As a result of the secondary emission of electrons from the plate of a tetrode and the collection of these electrons by the screen grid, considerable power is lost. The *pentode,* having a suppressor grid between the screen grid and plate, was introduced to control the secondary emission of electrons from the plate. Then, because there is considerable harmonic distortion in the pentode, the *beam-power pentode* was introduced. It has the efficiency and sensitivity of a pentode and the low distortion of a triode. In this tube the electrons are forced to pursue definite paths by proper focusing.

The present trend in tube design is to make specialty types. These tubes are either of the multielectrode type intended to give optimum performance in particular applications or of the multiunit type, which combines the functions of several tubes in a single envelope. The primary justification for these tubes seems to be space and economy considerations.

Then there are a multitude of special tubes such as the electron-ray tubes, gas tubes, cathode-ray tubes, X-ray tubes, and secondary-emission tubes. A detailed treatment of these tubes is beyond the scope of this chapter.

This chapter will be devoted to a treatment of the more important types of tetrodes, pentodes, and beam-power tubes, with only a brief treatment of the specialty types of tubes.

19.1. The Tetrode with Screen Grid. A triode plate serves two functions: (1) to affect the field in the tube, and (2) to collect electrons. In a tetrode the field is controlled by the screen grid. This leaves to the plate only one function, that of collecting electrons.

Since the elements in a vacuum tube are metallic conductors insulated from each other, there exist interelectrode capacities due to the electro-

static fields between the electrodes. These capacities are bilateral in nature and exist regardless of the electron action in the vacuum space. This destroys the desired unilateral properties between certain electrodes in the vacuum tube. It is possible by external neutralizing circuits to balance out this effect, but it is usually more desirable to minimize the effect by electrostatic shielding within the tube. In the tetrode, the shielding takes the form of a screen grid mounted between the control grid and the plate. In practice, the control-grid-to-plate capacity is reduced to the order of 10 $\mu\mu$f to 0.01 μf or less by the screen grid.

The screen grid, in addition to its desirable shielding properties, is operated at a positive potential, thus supplying an electrostatic force pulling electrons from the cathode to the plate. The cathode and control grid perform the same functions as in the triode. The screen grid supplies the accelerating potential for the electrons but does not collect very many of them, owing to the comparatively large space between grid wires. This means that the variations in plate voltage have very little effect on the plate current, resulting in a high plate resistance. Mathematically, this is readily seen by inspection of Eq. (18.2*a*) developed in the last chapter for the plate resistance. For a multielement tube it can be written, in partial-differential form,

$$r_p = \frac{\partial E_b}{\partial I_b} \tag{19.1}$$

where r_p = dynamic plate resistance, ohms
∂E_b = incremental change in plate voltage corresponding to ∂I_b
∂I_b = incremental change in electron plate current when all other parameters are held constant

The control-grid voltage of the tetrode is just as effective in varying the plate current as in the triode; hence the mutual conductance remains essentially unchanged. For the multielement tube it can be written [see Eq. (18.3*a*)]

$$g_{m1} = \frac{\partial I_b}{\partial E_{c1}} \tag{19.2}$$

where g_{m1} = mutual conductance of grid 1, mhos
∂I_b = incremental change in electron plate current corresponding to ∂E_{c1}
∂E_{c1} = incremental change in grid-1 voltage when all other parameters are held constant

The mutual-conductance effect of other grids can be secured by changing the numerical subscripts in this equation. The subscript refers to the number of the grid counting from the cathode to the plate.

By definition of the amplification factor [see Eq. (18.1*a*)], its value for a multielement tube is

$$\mu_1 = -\frac{\partial E_b}{\partial E_{c1}} \tag{19.3}$$

where μ_1 = amplification factor of grid 1
∂E_b = incremental variation in plate voltage corresponding to ∂E_{c1}
∂E_{c1} = incremental variation in grid-1 voltage when all other parameters are held constant

Here also, the amplification effect of the other grids can be secured by changing to the proper subscript. From this analysis it follows that a screen-grid tube can be designed to have the same mutual conductance as an equivalent triode, but it will have a very much higher plate resistance and amplification factor. By proper design it is possible also to make the mutual conductance high. The low control-grid-to-plate capacity makes it possible to realize the high amplification with good stability; hence the tetrode is especially desirable for voltage and power amplification.

The most serious limitation of the tetrode is the generation of secondary electrons at the plate caused by the accelerated electrons passing through the screen grid and striking the plate. When the plate voltage is lower than the screen-grid voltage, these secondary electrons splashed out of the plate are attracted back to the screen grid. If the *secondary emission* of electrons from the plate exceeds the number of primary electrons striking the plate, the external plate current will be decreased, giving a *negative resistance characteristic* due to the negative plate-current effect. Advantage is taken of this effect, for example, in the dynatron oscillator. Where the effect is undesirable, the pentode tube can be used to permit wider variations in plate current and plate voltage.

Example 19.1. Write down the plate resistance, mutual conductance, and amplification factor for the space-charge tetrode illustrated in Fig. 19.1*b*.

Solution

$$r_p = \frac{\partial E_b}{\partial I_b} \qquad Ans.$$

$$g_{m2} = \frac{\partial I_b}{\partial E_{c2}} \qquad Ans.$$

$$\mu_2 = -\frac{\partial E_b}{\partial E_{c2}} \qquad Ans$$

It will be noted that the only change as compared to Eqs. (19.1), (19.2), and (19.3) is in the subscript.

19.2. Tetrode Static Plate Characteristics. A family of static plate characteristics is plotted for a typical tetrode in Fig. 19.2. The solid

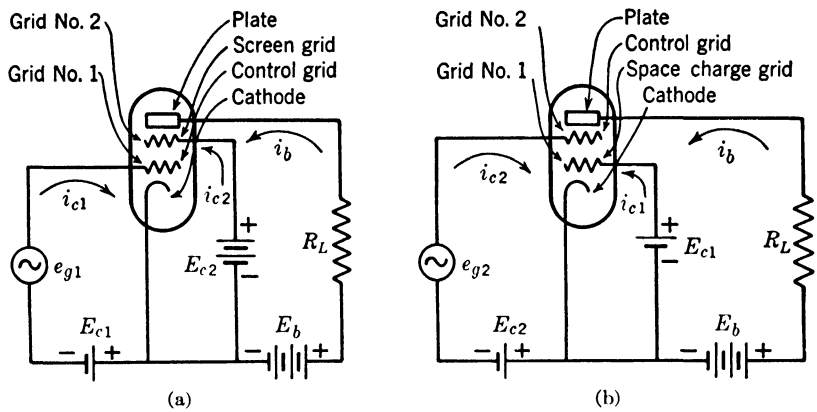


FIG. 19.1. Typical tetrode tube connections.

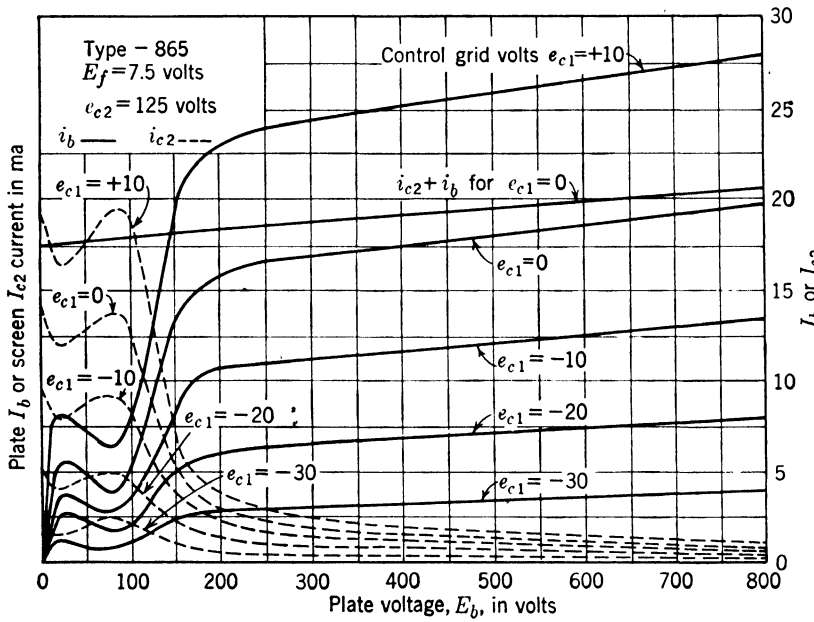


FIG. 19.2. Plate characteristic curves of a tetrode.

curves give the plate current, and the dashed curves give the screen current. The sum of the screen-grid current and plate current is practically a constant, as will be noted for the case shown when the control-grid voltage is zero ($i_{c2} + i_b$, for $e_{c1} = 0$). As the plate voltage is raised from zero, electrons passing through the screen are collected by the plate, giving a sharp rise in electron plate current. The velocity of the electrons striking the plate increases rapidly with plate voltage, and when

e_b in this case is about 25 volts, the electron velocity is sufficiently high to produce appreciable secondary emission. These secondary electrons are drawn to the screen, causing the electron plate current to drop off rapidly until the plate voltage has reached approximately 75 volts. Secondary-emission electron current rapidly drops above 75 volts plate voltage, until above approximately 200 volts the electron plate current becomes practically independent of plate voltage. For stable operation, this tube should be operated at plate voltages above 150 volts.

Example 19.2. Determine the plate resistance of the Type 865 tetrode by using the curves of Fig. 19.2, for $e_{c1} = -10$ volts and plate voltages between 300 and 700 volts.

Solution. Referring to Fig. 19.2 and approximating Eq. (19.1), we get

$$r_p = \frac{\partial E_b}{\partial I_b} \doteq \frac{600 - 300}{0.0125 - 0.01125} = \frac{300}{0.00125}$$
$$= 200{,}000 \text{ ohms} = \text{plate resistance} \qquad \textit{Ans.}$$

Example 19.3. If the plate voltage of the Type 865 tetrode of Example 19.2 is held constant at 600 volts and the bias is varied from 0 to -10 volts, determine the mutual conductance and amplification factor.

Solution. Substituting in Eq. (19.2), the approximate solution for the mutual conductance is

$$g_{m1} = \frac{\partial I_b}{\partial E_{c1}} \doteq \frac{0.0185 - 0.0125}{0 - 10} = \frac{0.0065}{10}$$
$$= 650 \ \mu\text{mho} \qquad \textit{Ans.}$$

And the amplification factor is [see Eq. (18.4)]

$$\mu_1 = r_p g_{m1} = (200{,}000)(650 \times 10^{-6}) = 130 \qquad \textit{Ans.}$$

19.3. Tetrode with Space-charge Grid. The negative space charge around the cathode can be neutralized by connecting the inner grid, that is, grid 1, to a small positive voltage; thus the electrons emitted by the cathode are readily controlled by grid 2. Grid 1 operating under this condition is known as the *space-charge grid.*

The advantages of the space-charge tetrode in comparison to the triode are

1. Lower plate-voltage operation
2. Higher mutual conductance
3. Lower plate resistance

The disadvantages as compared to the triode are

1. Relatively high space-charge grid current
2. Excessive curvature of the characteristic curves, which makes the output nonlinear

Since pentodes have superior properties, the space-charge tetrode is seldom used.

19.4. The Pentode with Suppressor Grid. For most applications the unstable portion of the tetrode characteristics, caused by secondary-emission electron current flowing from the plate to the screen grid, is

undesirable. The suppression of the secondary electron current can be accomplished by adding another grid between the screen grid and plate and connecting it to the cathode, as shown in Fig. 19.3*a*. This suppressor

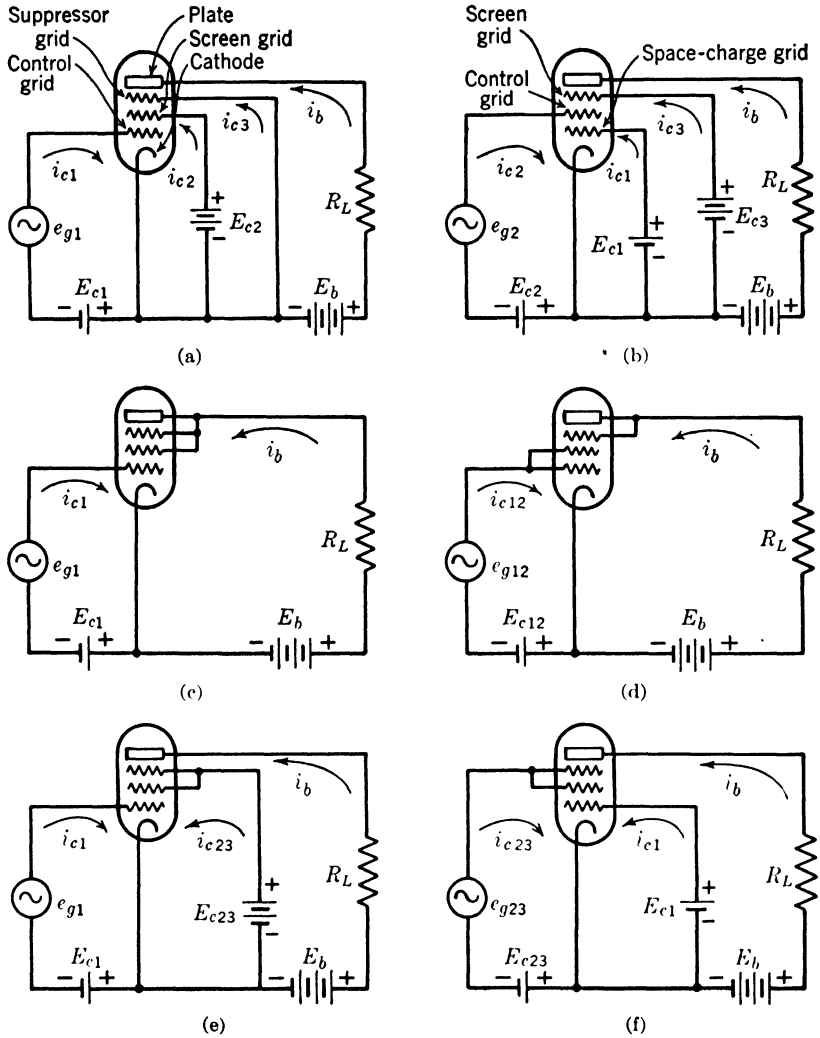


FIG. 19.3. Typical pentode tube connections.

grid creates a low potential field that very few of the secondary-emission electrons from the plate are able to penetrate. The screen grid is usually raised to a higher positive potential in the pentode in order to give the electrons from the cathode sufficient velocity to penetrate the field of the suppressor grid and reach the plate.

The shielding of the plate from the cathode is better than in the tetrode, with the result of a still higher plate resistance and amplification factor along with average values of mutual conductance. These tube constants have the same significance in a pentode or tetrode as in the prototype triode when the tubes are used as amplifiers. There are two general classifications of pentode amplifier tubes. The *power-output pentode* is designed to give high power output with low values of grid driving voltage while the *voltage-amplifier pentode* is designed to give high voltage

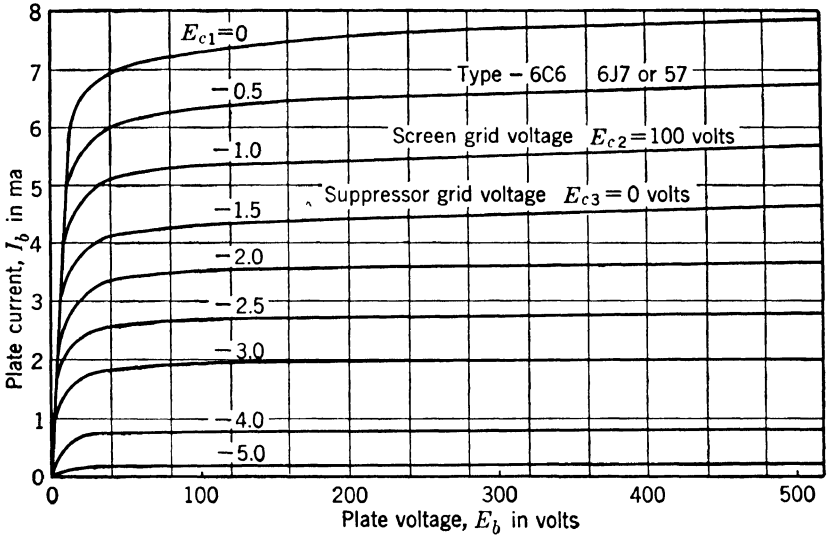


FIG. 19.4. Static plate characteristic curves of a typical pentode Type 6C6, 6J7, or 57. Screen grid voltage E_{c2} = 100 volts; suppressor grid voltage E_{c3} = 0 volts.

amplification at moderate values of plate voltage. The pentode in this respect is superior to the tetrode because the plate-voltage swing can be made much larger; in fact, the plate voltage may drop considerably below the screen voltage without serious loss in signal-gain capabilities.

19.5. Pentode Static Plate Characteristics. The static plate-characteristic curves of a typical voltage-amplifier pentode are shown in Fig. 19.4. The suppressor-grid action is made evident by the absence of the negative slopes of the plate-current curves. It is becoming more apparent that ideal static plate characteristics for an amplifier would be straight, parallel, and equidistant for all values of plate voltages. The bending of the characteristic at low voltages is the result of nonuniformity of the field in the plane of the suppressor grid. This is even more noticeable in power-output pentodes.

19.6. The Pentode with Other Connections. So far, the pentodes treated have had the suppressor-grid–screen-grid connection shown in

Fig. 19.3*a*. An obvious connection, as shown in Fig. 19.3*b*, is to use the first grid as a space-charge grid with the second grid as a control grid and the third grid as the screen grid. This makes a screen-grid tetrode with the addition of a space-charge grid, which permits the use of lower plate voltages. To go one step farther, a suppressor grid could be added to the tube in Fig. 19.3*b*, which would give a six-element tube with each element performing a definite function.

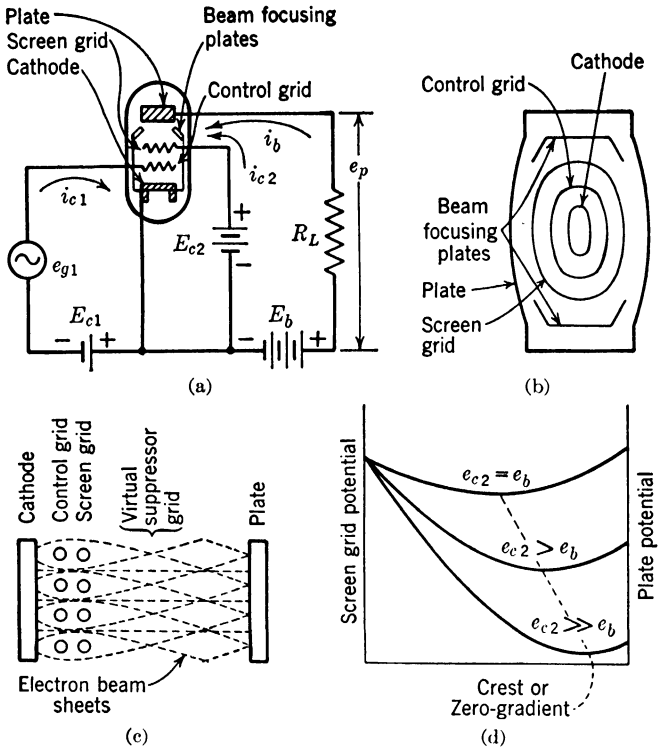


FIG. 19.5. Circuit construction and characteristics of a beam power pentode.

Figures 19.3*c* and *d* show how the electrodes of a pentode can be combined to give either a medium- or high-μ triode connection. A screen-grid or space-charge tetrode connection can be secured from a pentode if the electrodes are connected as shown in Figs. 19.3*e* and *f*. This by no means limits the number of connections of a pentode, some of which may have definite advantages for special applications.

19.7. Beam-power Pentode. The limitation in power-output pentodes is primarily due to third-harmonic distortion caused by the curvature of the plate characteristics at low plate voltages. At low plate voltages, the suppressor grid becomes so effective that it turns many primary electrons

back to the screen before they reach the plate. This is especially true along electron paths that intersect the suppressor-grid wires. This means that ideal results cannot be achieved by a mechanical suppressor-grid structure but can be achieved if there is sufficient electron density between the screen grid and plate to make the potential curves bend downward, as shown in Fig. 19.5*d*. The beam-power pentode developed by Schade meets these requirements.

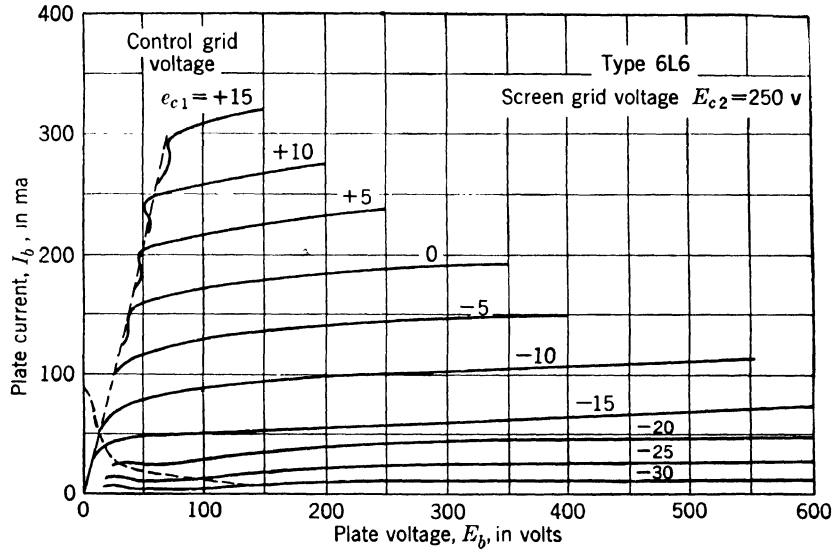


FIG. 19.6. Static characteristic curves of a typical beam power pentode.

The beam-power pentode is designed to meet the following requirements:

1. Sufficient electron density between the screen grid and plate to give a crest (or zero-gradient) point as shown in Fig. 19.5*d*, which acts like a suppressor grid, forcing secondary electrons back to the plate
2. Uniform electron density between the screen grid and plate
3. Proper location of crest between the screen grid and plate
4. Low screen-grid current

These requirements are met as follows:

1. Sufficient electron density is secured by the special contour design of the cathode, grids, and plate as shown in Fig. 19.5*b*.
2. Uniform electron density is obtained by confining the electrons to beams by the beam-focusing plates; this makes the suppressing field continuous and much more complete.
3. Proper location of the crest is the result of sufficient spacing between the screen grid and plate with the proper adjustment of the beam angle.

4. Low screen grid current and more uniform beam sheets result from placing the screen wires in the electronic shadow of the control grid, as shown in Fig. 19.5*c*.

Using a large cathode surface, with the advantages of low grid and plate operating temperatures, gives a tube of small size for its power-handling capacity. The output-voltage operating range is increased to make the *power sensitivity* ($\sqrt{P_{ac}}/E_g$) greater than that for any other type of receiving tube.

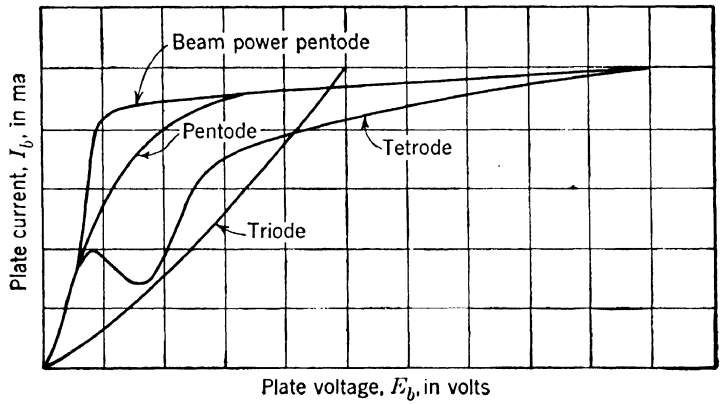


FIG. 19.7. Comparison of typical tube characteristics.

19.8. Beam-power-pentode Static Plate Characteristics. The static plate-characteristic curves of a typical beam-power pentode are given in Fig. 19.6. The general shape of the curves is similar to that for pentodes with a suppressor grid, with the exception that the linear portion extends to lower values of plate voltage and drops very abruptly.

19.9. Comparison of Static Plate Characteristics. In Fig. 19.7, a comparison of typical tube characteristics shows several important steps in the advancement of the art. The triode has a low plate resistance and amplification factor along with bilateral properties. By adding a screen, the plate resistance and amplification factor were made high, but over an appreciable low-voltage range the tube is unstable. By adding the suppressor grid, the plate resistance is made even higher, owing to the added shielding, and the tube is stable over the complete range. The operating range (linear portion of curve) of the pentode can be increased by removing the physical suppressor grid and replacing it by a virtual grid.

19.10. Variable-μ Pentode. The point at which the plate current becomes zero is known as the *cutoff point*. For linear characteristic curves, the bias for cutoff is

$$e_c = -\frac{e_b}{\mu} \tag{19.4}$$

where e_c = total instantaneous grid voltage, volts
e_b = total instantaneous plate voltage, volts
μ = amplification factor

The use of a nonuniform control-grid structure, as shown in Fig. 19.8*a*, will cause the μ to vary along the structure in such a way that the plate current is cut off where the grid mesh has close spacing and is not cut off as

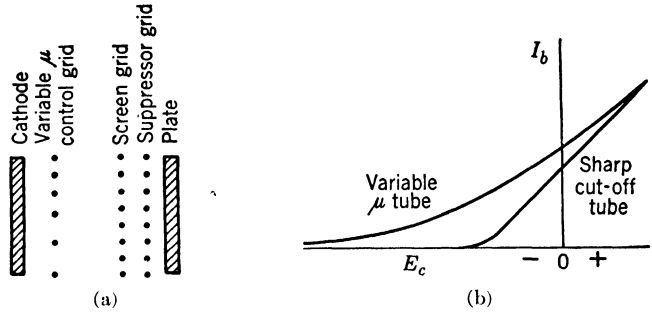


FIG. 19.8. Construction and characteristics of a variable-μ tube.

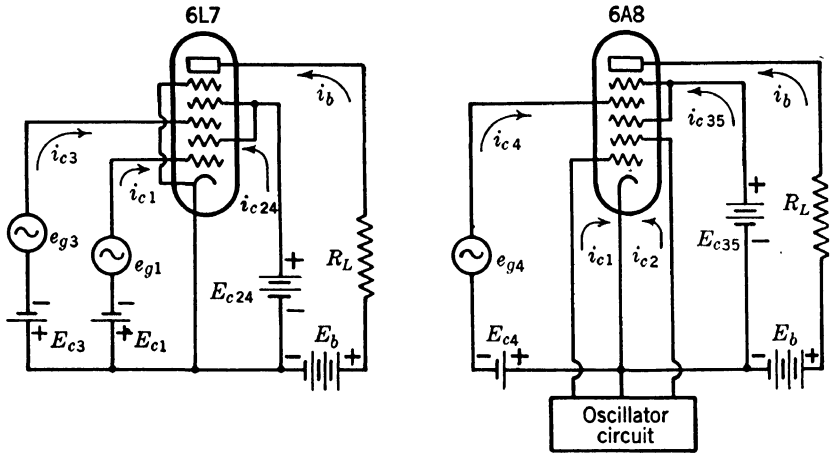
quickly where the grid mesh has wider spacing. This results in a gradual curvature in the E_c-I_b characteristic, as shown in Fig. 19.8*b*.

Many applications have been found for this type of tube characteristic. One of the most widely used applications is in the automatic volume control (avc) of modern radio receivers. Owing to the low curvature of the characteristics of this tube, weak signals are amplified on the steep portion of the curve (low bias), while strong signals are amplified on the portion of the curve having a small slope.

Cross talk is minimized by the use of the variable-μ tube. Since the change in slope is so gradual, a strong modulated carrier swinging the grid will have little modulating effect upon the desired signal, which is being amplified. It is the change in the slope of the curve that gives the nonlinear characteristic needed for applications such as modulation. Since the change in the slope of the variable-μ tube is so gradual, the cross-modulation effects are minimized.

19.11. Pentagrid-mixer Amplifier. The pentagrid-mixer amplifier, as shown in Fig. 19.9*a*, contains five grids, in addition to the cathode and plate. This tube is designed with two separate control grids shielded from each other, which permits each control grid to act independently on the electron stream. One common application of this tube is in superheterodyne receivers to mix the local-oscillator frequency with the incoming signal to produce an intermediate frequency in the output. In this

application, the incoming r-f signal is fed into grid 1, which has a variable-μ characteristic suited for a-v-c bias control. The local-oscillator voltage is applied to grid 3, which has a relatively large μ and a sharp cutoff. Grid 3 is electrostatically shielded from the other elements in the tube by grids 2 and 4, which are connected internally and also act as a screen grid to accelerate the electron stream. Grid 5 acts as a suppression grid to make the output plate characteristics like those of a pentode. The output circuit in this application is tuned to the intermediate frequency, which is either the difference or the sum of the input



(a) (b)

FIG. 19.9. Typical pentagrid, or heptode, tube connections.

frequencies. The inherent design of this tube makes it useful in a number of applications, such as in high-gain r-f or intermediate-frequency pentode amplifiers, and compressor or expander amplifiers.

19.12. Pentagrid Converter. The pentagrid converter, as shown in Fig. 19.9*b*, contains five grids in addition to the cathode and plate. This tube is designed to perform simultaneously the functions of an oscillator and a mixer tube like that used in superheterodyne receivers. In this application, grids 1 and 2, along with the cathode, are connected to an external circuit, as shown in Fig. 19.9*b*, to act as a triode oscillator. The grid of the triode oscillator is grid 1, and the plate is grid 2. Since the plate of the oscillator is in the form of a grid, an electron stream passes on through, owing to the field set up by the screen grids 3 and 5. These grids, which are connected internally, also act as a shield for grid 4. The r-f signal voltage is applied to grid 4 to modulate the electron stream, which is already modulated by the oscillator, thus producing a difference or sum frequency, which is removed from the plate circuit as the intermediate frequency.

The pentagrid converter has the merit of simplicity in acting as an oscillator and mixer, but since the shielding is not complete, its operation is not very satisfactory at the higher frequencies. For high-frequency operation, the trend is to use a separate oscillator and a pentagrid mixer.

19.13. Multiunit Tubes. Multiunit tubes combine in one envelope more than one distinct set of tube elements. The number of such combinations that can be devised is almost unlimited. Typical examples are the diode-pentode, the triode-pentode, and the diode-triode-pentode. For push-pull circuits, duplex-diode, twin-triode or twin-pentode tubes are very common. Another form of multiunit tube is the coupled triode, which contains two triodes, one of which is a voltage amplifier directly coupled with an output power amplifier, all within the same envelope.

In the design of equipment where economy and space are at a premium, one should refer to the tube manufacturer's vacuum-tube manual for possible useful combinations having the proper ratings.

19.14. Equivalent A-c Plate Circuits. The equivalent a-c plate circuit for a triode, as developed in the last chapter, is known as a constant-voltage-generator type of network. Owing to the high values of amplification factor and plate resistance in most multielement tubes, it is simpler to use the constant-current-generator type of equivalent circuit shown in Fig. 19.10*b*. To show that these circuits give the same results, the a-c plate current i_2, as given in the last chapter [compare Eq. (18.13)] and shown in Fig. 19.10*a*, is

$$i_p = \frac{\mu e_g}{r_p + R_L} \tag{19.5}$$

Multiplying both sides by R_L gives the voltage across R_L as

$$i_p R_L = \frac{\mu e_g R_L}{r_p + R_L} \tag{19.6}$$

Since $\mu = g_m r_p$, we can substitute for μ in Eq. (19.6) and get

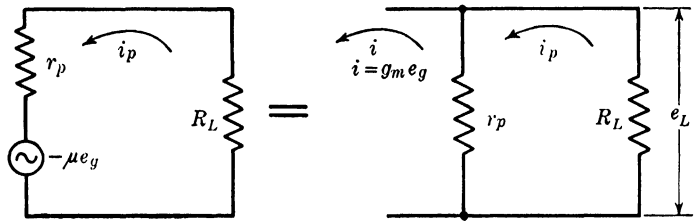
$$i_p R_L = g_m e_g \left(\frac{r_p R_L}{r_p + R_L} \right) \tag{19.7}$$

An inspection of this equation and Fig. 19.10*b* reveals that the current flowing through the parallel-circuit resistance must be

$$i = g_m e_g \tag{19.8}$$

where i = current flowing from parallel circuit as shown in Fig. 19.10*b* (instantaneous value of a-c current)
g_m = mutual conductance
e_g = a-c grid voltage (instantaneous value of voltage)

If there is more than one grid in the tube, the one causing the current flowing to the parallel circuit can be designated by subscripts to g_m and e_g. If more than one grid is causing plate current, then the superposition theorem can be applied to secure the total resulting alternating current in the plate circuit.



(a) (b)

FIG. 19.10. Alternating-current equivalent plate circuits.

Referring to Fig. 19.10*b* the a-c current i flowing in the parallel circuit produces an a-c load voltage e_L, thus

$$e_L = i \frac{r_p R_L}{r_p + R_L} \tag{19.9}$$

where e_L = instantaneous a-c component of load voltage, volts
i = instantaneous a-c component of parallel circuit current, amp

If effective values are used instead of instantaneous values, Eq. (19.9) can be written

$$E_L = I \frac{r_p R_L}{r_p + R_L} \tag{19.10}$$

Example 19.4. A pentode having a plate resistance of 0.6 megohm and a mutual conductance of 1,500 μmhos is connected to a load resistance of 0.5 megohm. Determine the effective value of current flowing to and the effective voltage across the equivalent parallel circuit when the effective grid voltage is 0.2 volts.

Solution. By placing effective values in Eq. (19.8), we get

$$I = g_m E_g$$

and from the problem

$$I = (1{,}500 \times 10^{-6})(0.2)$$
$$= 0.03 \text{ ma} \qquad \textit{Ans.}$$

The voltage across the load will be

$$E_L = I \left(\frac{r_p R_L}{r_p + R_L} \right)$$
$$= (0.0003) \left[\frac{(0.6 \times 10^6)(0.5 \times 10^6)}{(0.6 \times 10^6) + (0.5 \times 10^6)} \right]$$
$$= 8.182 \text{ volts} \qquad \textit{Ans.}$$

Summary

A tetrode is a four-element tube similar to a triode, except that an additional grid is introduced between the control grid and the anode. This additional grid,

which is known as the *screen grid*, is operated at a positive potential with respect to the cathode, and serves as an electrostatic shield between the anode and the control grid.

Both the dynamic plate resistance r_p and the amplification factor μ of a tetrode are much higher than the corresponding values for a triode. The mutual conductance g_m is about the same.

The pentode is a five-element tube similar to a tetrode, except that an additional grid is inserted between the screen grid and the anode. The function of this additional grid, which is known as the *suppressor grid*, is to prevent secondary emission from the anode.

A beam-power pentode is a pentode in which, instead of using an actual suppressor grid between the screen grid and the anode, a virtual suppressor grid is formed by causing a space charge to exist in this region.

A variable-μ pentode is a pentode in which the amplification factor is a function of the control-grid voltage.

A pentagrid-mixer amplifier contains five grids in addition to the cathode and plate. This tube is designed with two separate control grids shielded from each other, which permits each control grid to act independently on the electron stream.

Exercises

19.1. List the principal types of amplifier tube derived as the art progressed, with their contributions.

19.2. Explain why the mutual conductance of a tetrode or pentode is essentially the same as for a triode.

19.3. Why does a space-charge tetrode have a sharp cutoff of plate current?

19.4. What is the primary difference between the static plate characteristics of a pentode as compared to those of a tetrode? What would be the shape of ideal static plate characteristics?

19.5. Suggest a possible use of the various elements in a hexode tube used as an amplifier. What would be some of its general characteristics?

19.6. What limits the power output of pentodes? How does the beam-power pentode remedy the defect found in pentodes containing a physical suppressor grid?

19.7. Define in equation form "power sensitivity." How is the screen current of a beam-power pentode made low?

19.8. Can Eq. (19.4) be used for a variable-μ pentode?

19.9. List four useful applications of the pentagrid-mixer amplifier.

19.10. What is the justification for multiunit tubes?

CHAPTER 20

CATHODE-RAY TUBES

Cathode-ray tubes are designed for a purpose totally different from that of the tubes previously discussed. In the cathode-ray tube, a beam of electrons is produced by a so-called *electron gun*, and this beam of electrons strikes a phosphorescent screen on the inner surface of one end of the tube, where it is visible as a glowing spot. This spot can be made to move about on the phosphorescent screen by deflecting the electron beam through either magnetic or electrostatic potentials. Because the electron beam is virtually weightless, it can be made to follow the applied potentials even though these vary at the rate of millions of cycles per

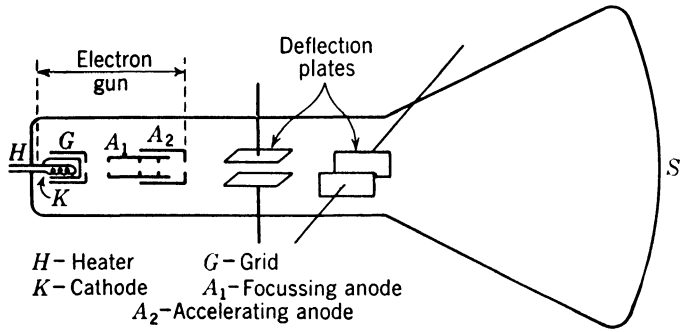


FIG. 20.1. Elements of a cathode-ray tube.

second. A cathode-ray tube is therefore a form of oscillograph, since it can be made to draw a picture of an electrical wave that is applied to it.

Besides their use in oscillographs, cathode-ray tubes perform many other important functions. They are the heart of television receivers and radar equipment. They are used for modulation monitors in radio transmitters, and in apparatus for checking the accuracy of carrier frequencies. In addition to these common uses, many special applications too numerous to list here also employ cathode-ray tubes.

20.1. The Electron Gun. The construction of a typical cathode-ray tube is shown in Fig. 20.1. The elements H, K, G, A_1, and A_2 constitute the electron gun, the purpose of which is to produce a sharply defined stream of high-speed electrons, which, in the absence of applied forces, would strike the phosphorescent screen S in the center and be visible as a small glowing spot there. The action of the electron gun is as follows:

The heater H supplies heat to the oxide-coated cathode K, which serves as a source for the electrons of the beam. Coaxial with the cathode and separated from it by a short distance is a metal cylinder pierced by a small hole. This cylinder is called the *grid* because its action is analogous with the action of the control grid in an ordinary vacuum tube, since the potential at which it is maintained with respect to the cathode controls the number of electrons that can pass through the center hole. This grid is customarily maintained at some potential negative with respect to the cathode, and the more negative it is made, the fewer electrons can pass.

The anode A_1 is maintained at a positive potential with respect to the cathode and attracts the electrons that escape through the hole in the grid. A_1 is a metal cylinder with several plates, or baffles, which are also pierced with holes in line with the hole in the grid. Many of the electrons that escape from the cathode through the grid are collected by the anode A_1, but some of the electrons pass directly through the holes in the baffles.

In the absence of the second anode A_2, no electrical gradient would exist beyond the anode A_1, and the electrons that had passed entirely through A_1 would, in the absence of A_2, continue on indefinitely until stopped by the interposition of some physical body such as the screen S. This beam would, however, have a tendency to diverge, owing to the mutual repulsion of the elements of the electron beam, and in order to get a well-defined spot on the screen, the second anode A_2 is used. By applying the proper voltage between anodes A_1 and A_2, the electron beam can be made to give a very small, very intense spot on the screen S. The more positive the grid G is made with respect to the cathode K, the greater the number of electrons that will pass through the grid, and consequently, the brighter will be the spot on the screen.

Since electrons are negatively charged particles, the screen would very quickly acquire a negative charge of such an intensity that it would repel the electron beam if means were not provided to remove this charge. This is accomplished by taking advantage of the secondary-emission characteristic of the screen coating. The electrons striking the screen have a very high velocity, and for every electron that strikes the screen, one or two electrons are knocked loose from the screen material by secondary emission. The direction of these electrons as they leave the screen is back into the tube, and the walls of the tube right up to the screen are coated with a conductive material such as Aquadag. This conductive coating is connected to the second anode A_2 and collects the electrons that are emitted from the screen and carries them off just as though they had been collected by the second anode in the first place instead of passing through it. The fact that the inner coating of the tube is at the same potential as the second anode causes no difficulty in

the electric-field distribution of the electron gun, since the conductive coating is distributed over the entire inner surface of the tube between the second anode and the screen, and is symmetrical with respect to the central axis of the gun.

20.2. Deflection Methods. As was previously stated, two methods are used to deflect the electron beam from its central position on the screen. The more common of these is the *electrostatic deflection method,* which consists in setting up an electrostatic field whose gradient is perpendicular to the normal direction of the electron beam. If a pair of

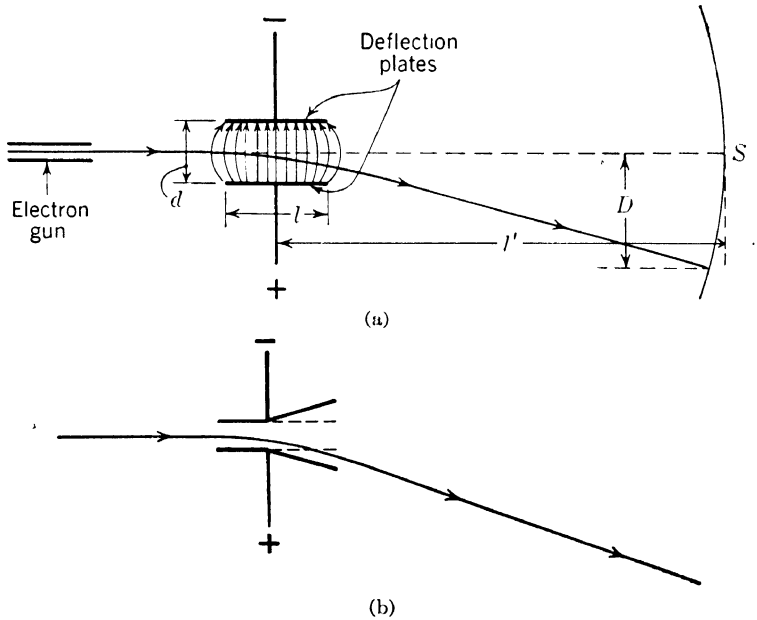


FIG. 20.2. Electrostatic deflection method.

metal plates is located on each side of the central axis of the tube and connected to the second anode, as shown in Fig. 20.2, and if a voltage source is connected between the plates, the electron beam—since it consists of negatively charged particles—will be repelled by the negative plate and attracted to the positive plate, and will consequently be deflected from its normal central position and will strike the screen in a new spot.

In cathode-ray tubes that employ electrostatic deflection, the plates are normally not made parallel, since such positioning would limit the angle through which the electron beam could be made to move. If too great a voltage were applied between the deflecting plates, the beam would strike the edge of the deflection plate closest to the screen, and all the

electrons in the beam would be collected by this plate, since it is at the same potential as the second anode. If the deflection plates are spaced further apart to permit a greater angle of deviation of the beam without its striking the plates, the voltage that it is necessary to apply between the plates to secure a given deflection of the beam will increase.

Since, naturally, it is desirable to obtain the greatest motion of the beam for the smallest applied deflection voltage, another scheme is employed that permits a much wider angle of deviation without at the same time greatly decreasing the deflection sensitivity of the tube. This consists in the use of deflection plates that are inclined at an angle with the beam, as shown in Fig. 20.2*b*. With such plates it is possible to obtain as great an angle of deflection as desired without at the same time impairing the deflection sensitivity of the tube.

The use of only one pair of deflection plates will permit the spot on the screen to be moved in one direction only, that is, in a line perpendicular to the plane of the plates. By employing two sets of deflecting plates at right angles to each other, as shown in Fig. 20.1, it is possible to move the spot to any point on the screen by applying the proper voltages to both sets of plates simultaneously. One set of plates moves the spot up and down on the screen; these are called the *vertical-deflection plates*. The other set of plates moves the spot back and forth across the screen; these are called the *horizontal-deflection plates*.

The other method of deflection, which is widely used in television, consists in the use of a magnetic field at right angles to the direction of the electron beam. This field is applied by means of a pair of deflecting coils, as shown in Fig. 20.3*a*. The coils are located outside the tube, as distinguished from the electrostatic-deflection plates previously considered, which are almost invariably incorporated into the tube structure. Current flowing through the deflection coils produces a magnetic field, as shown in Fig. 20.3*b*, and the electron beam, by virtue of the velocity of the particles that compose it, is bent as it passes through the field.

It can be shown that, using electrostatic deflection as shown in Fig. 20.2, the deflection D that will be realized for a given deflecting voltage E_d will be given by

$$D = \frac{l'lE_d}{2dE_a} \tag{20.1}$$

where D = deflection on screen, cm
l' = length of beam from center of deflecting plates to screen, cm
l = length of deflecting plates in direction of beam, cm
d = distance between deflecting plates, cm
E_d = deflecting voltage between plates, volts
E_a = beam voltage, volts

The name *deflection sensitivity* is given to the ratio of D/E_d, and this ratio is generally designated by S. Using this notation, Eq. (20.1) then becomes

$$S = \frac{D}{E_d} = \frac{l'l}{2dE_a} \qquad \text{cm/volt} \tag{20.2}$$

S is therefore proportional to the length of the deflection plates and the distance from the deflection plates to the screen, and inversely proportional to the distance between the deflection plates and the potential on the accelerating anode.

This equation was derived under the assumption that the electrostatic field between the deflection plates is perpendicular to the plates at all points and does not extend beyond them. This, of course, is not actually the case, since there will be a curved fringing field existing beyond the ends of the plates, as shown in Fig. 20.2*a*. This fringing field makes the effective length of the deflection plates somewhat longer than the actual length; moreover the effective length will be a function of d, the distance between the plates. This result might have been expected, since it is fairly clear that the farther apart the deflection plates are, the farther will the fringing field extend. For plates separated by only an infinitesimal distance, there is virtually no fringing field. As a matter of fact, it can be shown that very correct results can be achieved using Eq. (20.2), if for l we substitute l'' where

$$l'' = l + d \tag{20.3}$$

Instead of rating cathode-ray tubes by their deflection sensitivity, they are often rated by their *deflection factor*, which is the reciprocal of the sensitivity. This is generally denoted by G, and from Eq. (20.2) this is given immediately as

$$G = \frac{1}{S} = \frac{2dE_a}{l'l} \qquad \text{volt/cm} \tag{20.4}$$

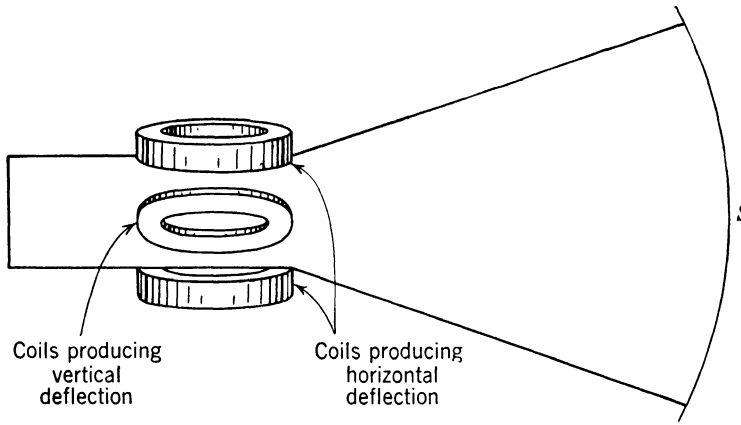
and the terms are defined as in Eq. (20.1).

The occurrence of E_a in these equations indicates that in order to maintain the sensitivity or the deflection factor of a cathode-ray tube constant, it is necessary to supply an accurately maintained voltage to the accelerating anode A_2, if accurate measurements are to be obtained with the tube.

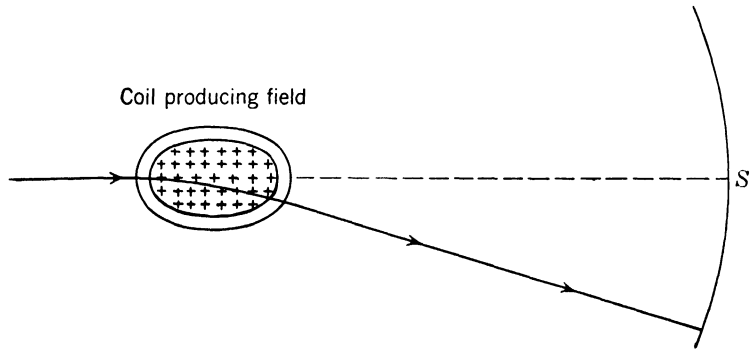
A somewhat different expression for the sensitivity is obtained when magnetic deflection is used. In this case the sensitivity is given in terms of centimeters per gauss instead of centimeters per volt, as in the case of electrostatic deflection. The magnetic field intensity in gausses is generally designated by B, and using this notation, the deflection sensitivity of a cathode-ray tube using magnetic deflection is given by

$$\frac{D}{B} = \frac{l'l}{\sqrt{E_a}} \frac{\sqrt{e}}{\sqrt{2m \times 10^9}} \quad \text{cm/gauss} \qquad (20.5)$$

where D = deflection, cm
B = magnetic field intensity, gausses
l' = length of deflection plates, cm
l = distance from deflection plates to screen, cm
e = charge of electron, coulombs
m = mass of electron, g
E_a = accelerating voltage, volts



(a) Use of coils for magnetic deflection.



(b) Path of an electron in a magnetic field (magnetic field into paper).

FIG. 20.3. Magnetic field deflection method.

Several differences will be noted immediately between Eqs. (20.5) and (20.2). Equation (20.5) contains the term e/m, which is the ratio of charge to mass of an electron. This factor does not enter in Eq. (20.2). The other outstanding difference is the fact that the deflection sensitivity for magnetic deflection varies inversely as the square root of E_a, whereas

using electrostatic deflection, the sensitivity varies inversely as E_a. This is a decided advantage in favor of magnetic deflection, since it means that the sensitivity will show considerably smaller variations with changes in the accelerating-anode potential than in the case of electrostatic deflection. The magnetic field deflection method is shown in Fig. 20.3.

20.3. Screen Materials. The material that is used in the screen determines the color of the spot generated by the impact of the beam electrons, the intensity of the spot, and the length of time it will glow after the beam electrons cease to strike, or in other words, the *persistence*. Screens of several different compositions are employed, depending on the use to which the tube is to be put. The most common material for screens is zinc orthosilicate, which in its natural form is called *willemite*. When a beam of electrons strikes a screen composed of this material, the color of the spot is a yellowish green, which corresponds to the color to which the human eye is the most sensitive. This material exhibits some degree of phosphorescence, which makes it unsuitable for use in applications where rapidly changing patterns are to be viewed on the screen.

To secure a screen with very short persistence, calcium tungstate and cadmium tungstate are used. These materials exhibit practically no phosphorescence or afterglow. These screens give a blue or violet spot, to which the eye is rather insensitive, but which makes them excellent for photographic purposes.

Zinc sulfide and zinc cadmium sulfide give screens a very long persistence. Such screens are useful for studying very rapid transients, which themselves persist for only a fraction of a second, but which can be seen on such a long-persistence screen for a number of seconds afterwards. These screens give a blue or blue-white spot.

20.4. Sweep Action in Cathode-ray Tubes. One of the chief uses of the cathode-ray tube is in the cathode-ray oscilloscope, a device that produces a visual representation of the wave form of an electrical voltage applied to one pair of the deflecting plates of the tube. If the wave form, amplitude, and frequency of this applied voltage are constant, it is possible to produce a stationary picture of it on the screen of the tube, by the application of the proper *sweep voltage* to the other pair of deflecting plates.

Suppose that a sinusoidal voltage were applied to the vertical-deflection plates only, no voltage being applied to the horizontal plates. The spot would move up and down on the screen of the tube (Fig. 20.4*b*), its deflection from the center at any instant being proportional to the instantaneous value of the applied voltage. Owing to the persistence of the screen material, and the persistence of vision of the observer, the latter would see a stationary vertical line on the screen.

If, however, the spot were drawn steadily across the screen from left to right at the same time that the signal was being applied (Figure 20.4*a*), the spot would trace out a path on the screen that would be an exact

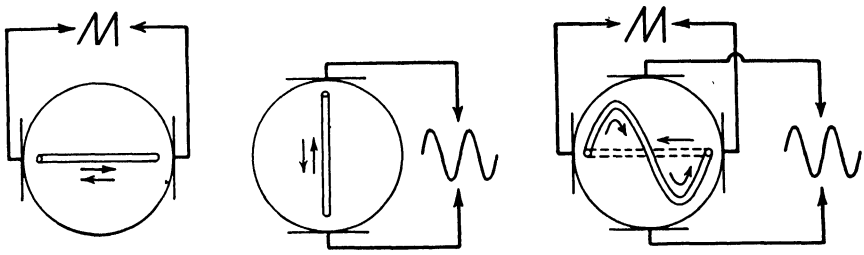


FIG. 20.4. How a cathode ray tube is used to present a signal.

picture of the wave form of the signal. If, furthermore, the spot were drawn steadily from left to right for a time just equal to the period of the applied signal, exactly one cycle of the signal would be traced out. If this process were repeated over and over, the observer would see an apparently stationary picture of the signal wave form on the screen of the tube, as a result of persistence of vision. In order to miss no part of the cycle, the spot must be brought back to its starting point each time very quickly as shown in Fig. 20.4*c*.

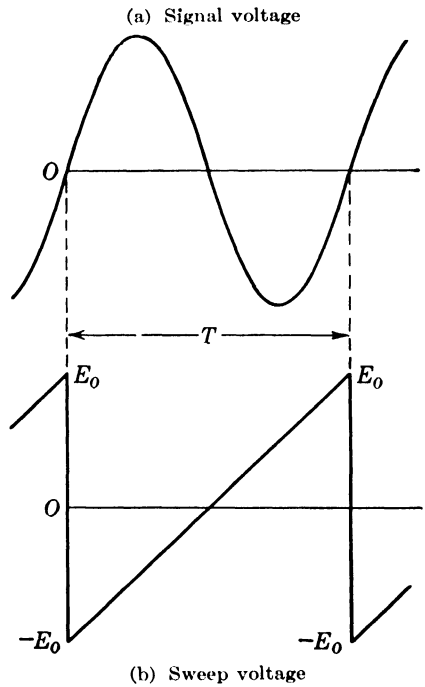


FIG. 20.5. Signal and sweep voltage in a cathode ray tube.

The foregoing requirements for a sweep voltage are met by the *saw-toothed* voltage shown in Fig. 20.5*b*. The maximum value of this voltage, E_0, must be sufficient to deflect the spot as far to the right of center as desired, and the minimum value $-E_0$ must likewise deflect the spot to the left of center. Starting from the value $-E_0$, which deflects the spot to the extreme left of the screen, the sweep voltage increases steadily until the value E_0 is attained. At this point the sweep voltage drops abruptly to $-E_0$, returning the spot to the left in a very short time.

The path of the spot from the extreme right to the original starting

point is called the *return trace.* Many cathode-ray oscilloscopes incorporate circuits that supply a negative pulse to the grid of the cathode-ray tube during the time of each return trace, thus cutting off the electron beam and thereby rendering the return trace invisible.

Summary

In the cathode-ray tube a beam of electrons is produced by an electron gun, and this beam of electrons strikes a phosphorescent screen on the inner surface on one end of the tube, where it is visible as a glowing spot. This spot can be made to move about on the phosphorescent screen by deflecting the electron beam through either magnetic or electrostatic potentials.

Using electrostatic-deflection methods, the deflection D that will be realized for a given deflecting voltage E_d will be given by

$$D = \frac{l'lE_d}{2dE_a}$$

where D = deflection, cm
l' = effective length of deflecting plates, cm
l = actual length of deflecting plates, cm
E_d = deflecting voltage, volts
d = distance between deflecting plates, cm
E_a = accelerating voltage, volts

The ratio D/E_d is called the *deflection sensitivity*, and is generally designated by S.

The reciprocal of the deflection sensitivity is called the *deflection factor*, and is generally denoted by G.

When magnetic deflection is used, the deflection sensitivity of a cathode-ray tube is given by

$$\frac{D}{B} = \frac{l'l}{\sqrt{E_a}} \sqrt{\frac{e}{2m \times 10^9}}$$

where D = deflection, cm
B = magnetic field intensity, gausses
l' = length of deflection plates, cm
l = distance from deflection plates to screen, cm
e = charge of electron, coulombs
m = mass of electron, g
E_a = accelerating voltage, volts

Exercises

20.1. What is the function of the electron gun in a cathode-ray tube?

20.2. How is the electron beam made visible in a cathode-ray tube?

20.3. How is the intensity of the electron beam varied?

20.4. What measures are taken to prevent the accumulation of charges on the walls of a cathode-ray tube?

20.5. What is meant by "deflection sensitivity"? "Deflection factor"?

20.6. What advantage does magnetic deflection have over electrostatic deflection?

20.7. In a certain cathode-ray tube employing electrostatic deflection, E_a = 1,000 volts, l = 3 cm, l' = 15 cm, and d = 1 cm. What is the deflection sensitivity?

20.8. Solve Exercise 20.7 using the expression for the corrected length l'' given in Eq. (20.3).

20.9. What wave form is used for a sweep voltage?

20.10. What is meant by the "return trace"?

APPENDIX

1. GREEK ALPHABET

Letters		Name	Pronunciation	Equivalent
Capital	Small			
A	α	Alpha	ăl′fȧ	a
B	β	Beta	bē′tȧ or bā′tȧ	b
Γ	γ	Gammą	găm′ȧ	g
Δ	δ	Delta	dĕl′tȧ	d
E	ϵ	Epsilon	ĕp′sĭ-lŏn	e (short)
Z	ζ	Zeta	zē′tȧ or zā′tȧ	z
H	η	Eta	ē′tȧ or ā′tȧ	e (long)
Θ	θ	Theta	thē′tȧ or thā′tȧ	th
I	ι	Iota	ī-ō′tȧ	i
K	κ	Kappa	kăp′ȧ	k, c
Λ	λ	Lambda	lăm′dȧ	l
M	μ	Mu	mū, mo͞o, or mü	m
N	ν	Nu	nū or nü	n
Ξ	ξ	Xi	zī or ksē	x
O	o	Omicron	ŏm′ĭ-krŏn or ō-mī′krŏn	o (short)
Π	π	Pi	pī or pē	p
P	ρ	Rho	rō	r
Σ	σ	Sigma	sĭg′mȧ	s
T	τ	Tau	tô or tou	t
Υ	υ	Upsilon	ūp′sĭ-lŏn	u, y
Φ	ϕ	Phi	fī or fē	ph
X	χ	Chi	kī, kē, or ĸē	kh, ch
Ψ	ψ	Psi	psē or sī	ps
Ω	ω	Omega	ō-mē′gȧ ō-mĕg′ȧ, or ō′mĕ-gȧ	o (long)

ăt, āte, operā;
nĕt, mē, pĭn, pīne;
nŏt, ôrb, nōte, fo͞o.l, ŭp, lūte, menū;
ĸ-ch, in German *ach*.

2. MATHEMATICAL SYMBOLS AND ABBREVIATIONS

Symbol	Name	Uses
1, 2, 3, 4, 5, 6, 7, 8, 9, 0	Arabic numerals	The numerals used in mathematics
∞	Infinity	A quantity larger than any number
.	Decimal point	A period used to point off numbers
$+$	Plus	Addition, positive
$-$	Minus	Subtraction, negative
$\pm$	Plus or minus	Addition or subtraction
$\mp$	Minus or plus	Subtraction or addition
$\times$	Multiply by	Multiplication
$a \cdot b$	a is multiplied by b	Multiplication
$(2 \cdot 1)(3 \cdot 4)$	2·1 multiplied by 3·4	Quantity multiplication
$\div$	Divided by	Division
a/b	a is divided by b	Division (fraction form), or per
$\frac{a}{b}$	a is divided by b	Division (fraction or ratio form)
$a:b$	a is to b	Ratio form of division
$=$	Equal to	Equality in equation form
$\equiv$	Identical to	Identity in equation form
$\doteq$	Approximately equal to	Approximate equality in equations (sometimes used for approaches)
$\neq$	Not equal to	Inequality in equation form
$<$	Less than	Inequality in equation form
$>$	Greater than	Inequality in equation form
$\leqq$	Equal or less than	Equality or inequality in equation form
$\geqq$	Equal or greater than	Equality or inequality in equation form
$\propto$	Proportional to	Proportionality used in equation form
::	Proportional to	Proportionality used in equation form
$\rightarrow$	Approaches	To express that a quantity approaches in equation form
$\perp$	Perpendicular	Geometry
$\parallel$	Parallel to, or in parallel with	Geometry, or parallel electrical circuits
$\angle$	Positive angle sign	Geometry, vector quantities
$\overline{\angle}$	Negative angle sign	Geometry, vector quantities
$\triangle$	Triangle	Geometry
$\odot$	Circle	Geometry
$\therefore$	Therefore	Geometry
#	Number	When placed before figures
%	Per cent	Placed after number expressing per cent
· · ·	Continued	To indicate the continuation of a number or series
π	Pi	$\pi = 3.1416 \cdot \cdot \cdot$ radians, the circumference of a circle divided by its diameter

Symbol	Name	Uses
ϵ	Epsilon	$\epsilon = 2.71828 \cdots$, the base of natural logarithms
$3!$ or $\underline{\lfloor 3}$	Factorial	To express the fact that a number is multiplied by all smaller numbers, thus $3! = \underline{\lfloor 3} = 3 \times 2 \times 1 = 6$
Σ	Summation	To add a series of terms
$\sum_{0}^{n}$	Summation from 0 to n	To add a series of terms from 0 to n
( )	Parentheses	A sign placed at both ends of a grouping
[ ]	Brackets	A sign placed at both ends of a grouping
{ }	Braces	A sign placed at both ends of a grouping
$\overline{\quad}$	Vinculum	A line placed over a grouping
$\surd$	Radical	Extract the root of a number
$\surd\overline{\quad}$	Radical and vinculum	Extract the square root of a quantity or number
$\sqrt[3]{A}$ or $A^{1/3}$	3 is the root of the quantity A	Expressing the root to be extracted, or $\frac{1}{3}$ is the power to which it must be raised
A^2	2 is the exponent of the quantity A	Expressing the power to which a quantity is to be raised
$\lvert A \rvert$	A has a magnitude only	To express only the magnitude of a quantity
$\mathbf{A}$ or $\dot{\mathrm{A}}$ or $\bar{\mathrm{A}}$	A is a vector	To express a quantity that has both magnitude and direction
A	A is a vector	When the magnitude is expressed $\lvert A \rvert$, the A can be used to express a vector quantity
A_x	x is a subscript	To identify the quantity A, as, for example, lying along the x-axis
j	Operator j	To express quantities in the direction of the y-axis. (In pure mathematics, i is used for this operator and is called an imaginary number.)
$^\circ$	Degree sign	To express angles in degrees
$'$	Minute sign	To express angles in minutes
$''$	Second sign	To express angles in seconds
$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	Determinate sign	To express the solution of simultaneous equations
Δx	Increment of x	Small changes, in calculus
dx	Differential of x	Differential, in calculus
∂x	Partial differential of x	Partial differential, in calculus
∇	Del	Differential operator, in differential calculus

Symbol	Name	Uses
$\int$	Integral	Integral calculus
$\int_b^a$	Integral between the limits of a and b	Integration between limits
$F(x)$ or $f(x)$	Function of x	Algebra or calculus
$J_n(x)$	Bessel function of the first kind of order n	Higher mathematics—Frequency modulation
$K_n(x)$	Bessel function of second kind of order n	Higher mathematics
Γ	Gamma function	Higher mathematics

Abbreviation	Name	Uses
ln, log	Logarithm	Express the natural and common logarithm of a quantity, respectively
log *or* $\log_{10}$	Logarithm to base 10	This is the common base and is usually written log
ln *or* $\log_\epsilon$	Logarithm to base ϵ	This is the natural base and is usually expressed ln (sometimes called "Napierian")
colog	Cologarithm	The logarithm of the reciprocal of a number
antilog	Antilogarithm	To express a number in terms of its logarithm (the reverse operation of taking the logarithm of a number)
sin	Sine of	Circular trigonometry
$\sin^{-1}$	Arc whose sine is	Antisine of, inverse of sine (circular trigonometry)
sinh	Hyperbolic sine of	Hyperbolic trigonometry
$\sinh^{-1}$	Inverse hyperbolic sine of	Hyperbolic trigonometry
cos	Cosine of	Circular trigonometry
$\cos^{-1}$	Arc whose cosine is	Anticosine of, inverse cosine of (circular trigonometry)
cosh	Hyperbolic cosine of	Hyperbolic trigonometry
$\cosh^{-1}$	Inverse hyperbolic cosine of	Hyperbolic trigonometry
tan	Tangent of	Circular trigonometry
$\tan^{-1}$	Arc whose tangent is	Antitangent of, inverse of tangent (circular trigonometry)
tanh	Hyperbolic tangent of	Hyperbolic trigonometry

Abbreviation	Name	Uses
$\tanh^{-1}$	Inverse hyperbolic tangent of	Hyperbolic trigonometry
cot	Cotangent of	Circular trigonometry
$\cot^{-1}$	Arc whose tangent is	Anticotangent of, inverse of cotangent (circular trigonometry)
coth	Hyperbolic cotangent of	Hyperbolic trigonometry
$\coth^{-1}$	Inverse hyperbolic cotangent of	Hyperbolic trigonometry
csc	Cosecant of	Circular trigonometry
$\csc^{-1}$	Arc whose cosecant is	Anticosecant of, inverse of cosecant (circular trigonometry)
csch	Hyperbolic cosecant of	Hyperbolic trigonometry
csch^{-1}	Inverse hyperbolic cosecant of	Hyperbolic trigonometry
sec	Secant of	Circular trigonometry
$\sec^{-1}$	Arc whose secant is	Antisecant of, inverse of secant (circular trigonometry)
sech	Hyperbolic secant of	Hyperbolic trigonometry
sech^{-1}	Inverse hyperbolic secant of	Hyperbolic trigonometry
vers	Versed sine	vers $\beta = 1 - \cos \beta$ (circular trigonometry)
covers	Coversed sine	covers $\beta = 1 - \sin \beta$ (circular trigonometry)
hyp	hypotenuse	Trigonometry
opp	opposite side	Trigonometry
adj	adjacent side	Trigonometry
L.C.D.	least common denominator	Algebra

3. ENGINEERING SYMBOLS AND ABBREVIATIONS

a. Greek alphabet symbols:

- α Angles, attenuation factor, radiation
- β Angles, wave length constant, radiation
- Γ Complex Hertzian vector
- γ Angles, propagation constant, conductivity, radiation, Euler's constant
- Δ Increments, variation
- δ Decrement, density
- ϵ Base natural logarithms, very small quantity, inductivity, permittivity
- η Efficiency, elliptic coordinate

Θ Angles, temperature
θ Angles, elevation angle, time constant, phase displacement
κ Boltzmann's constant, dielectric constant
Λ, λ Wave length of electric waves
μ Amplification factor, permeability, prefix micro
ν Frequency of radiation
ξ Elliptic coordinate
Π Hertz vector
π Circumference of a circle divided by its diameter
ρ Resistivity, specific resistivity, charge density
Σ Summation
σ Conductivity
τ angles, tangent vector, surface density
Φ Scalar velocity potential, magnetic flux
ϕ Angles, flux, power factor angle, true azimuth, electric potential
χ Magnetic susceptibilities
Ψ Angles, phase difference, scalar function
Ω Resistance in ohms
ω Resistance in ohms, angular velocity

b. English alphabet symbols:

A Area, current electromagnetic vector potential
a Dimensions, amperes
B Susceptance, magnetic inductance, magnetic flux density
b Susceptance
C Capacity, centigrade
c Constant, velocity of light
D Electric displacement, electrostatic flux density
d Distance, diameter
E Effective voltage, emf, electric field intensity
e Instantaneous voltage
F Force, electrostatic field intensity
f Frequency, force, farad
G Conductance measured in mho
g Conductance, acceleration due to gravity, gram
H Magnetic field intensity
h Height, Plank's constant, henry
I Effective current
i Instantaneous current
J Poynting's vector, current density vector, intensity of magnetization
j Operator
K Constant, dielectric constant, stacking factor
k Susceptibility, prefix "kilo-"
L Self inductance
l Length
M Mutual inductance, meters, induced polarization, prefix "mega-"
m Mass, prefix "milli-"
N Number of conductors or turns, Avogadro's number
n Number, velocity of rotation

O Origin of coordinate system
o Origin of coordinate system
P Power
p Sound pressure, instantaneous power
Q Quantity of electricity, ratio of reactance to resistance
q Electric charge
R Resistance, reluctance
r Resistance, radius
S Sensitivity, surface area
s Transconductance
T Absolute temperature, periodic time, absolute temperature, tension
t Time
U Energy
u Particle velocity
V Voltage, volume, reluctivity
v Velocity
W Energy
w Watts
X Coordinates, reactance
x Coordinates, reactance
Y Coordinates, admittance
y Coordinates, admittance
Z Coordinates, impedance
z Coordinates, impedance

c. Quantitative electrical abbreviations:

Capacitance

μμf	Micromicrofarad (one-millionth of a millionth part of a farad)
μf	Microfarad (one-millionth of a farad) NOTE: In some cases, mfd or mf are used instead of μf to indicate one millionth of a farad. It is always safe to assume that mfd or mf means microfarad and not millifarads.
f	Farad NOTE: The letter f is rarely used alone. It is preferable to spell the word out when it is used.

Current

μa	Microampere (one-millionth of an ampere)
ma	Milliampere (one-thousandth of an ampere)
a or amp	Ampere

Frequency

cps	Cycles per second
kc	Kilocycles per second
Mc	Megacycles per second

Inductance

μh	Microhenry (one-millionth of a henry)
mh	Millihenry (one-thousandth of a henry)
h	Henry

Power

μw	Microwatt (one-millionth of a watt)
mw	Milliwatt (one-thousandth of a watt)
w	Watt
kw	Kilowatt (one thousand watts)
Mw	Megawatt (one million watts)
kva	Kilovolt ampere—apparent power
kwhr	Kilowatt hour

Resistance

Ω	Ohms NOTE: In some cases ω is used to express ohms.
kΩ	Kilohm (one thousand ohms)
MΩ	Megohm (one million ohms) NOTE: In some cases Meg is used to express megohms.

Voltage

μv	Microvolt (one-millionth of a volt)
mv	Millivolt (one-thousandth of a volt)
v	Volt
kv	Kilovolt (one thousand volts)

***d.* Circuit Symbols:**

Admittance

Y	Admittance in mhos
B	Susceptance in mhos
B_C	Capacitive susceptance in mhos
B_L	Inductive susceptance in mhos

Constants

G	Conductance in mhos
C	Capacitance in farads
L	Inductance in henrys
M	Mutual inductance in henrys
R	Resistance in ohms
G	Conductance in mhos

Current

I	Effective (or rms) current in amperes
I_m	Maximum (or peak) current in amperes[1]
I_{av}	Average current in amperes[2]
i	Instantaneous current in amperes

Impedance

Z	Impedance in ohms
X	Reactance in ohms
X_C	Capacitive reactance in ohms
X_L	Inductive reactance in ohms
R	Resistance in ohms

Miscellaneous

Ant	Antenna
Gnd	Ground
Mod	Modulator
Spkr	Loud speaker

[1,2] See footnotes on next page.

Power supply

A	Filament power supply
B	Plate power supply
C	Grid-bias power supply

Tube

Fil	Filament
K or k	Cathode
H or h	Heater
G or g	Grid
S_g	Screen grid
P or p	Plate

Voltage

E	Effective (or rms) voltage in volts
E_m	Maximum (or peak) voltage in volts[1]
E_{av}	Average voltage in volts[2]
e	Instantaneous voltage in volts

e. Abbreviations:[3]

Current

ac	Alternating current
dc	Direct current

Frequency

af	Audio frequency
if	Intermediate frequency
rf	Radio frequency
trf	Tuned radio frequency
bf	Beat frequency
bfo	Beat-frequency oscillator
lf	Low-frequency (30 to 300 kc band)
mf	Medium-frequency (300 to 3,000 kc band)
hf	High-frequency (3,000 kc to 30 Mc band)
vhf	Very-high-frequency (30 to 300 Mc band)
uhf	Ultra-high-frequency (300 to 3,000 Mc band)

Miscellaneous

cw	Continuous wave
icw	Interrupted continuous wave
am	Amplitude modulation
fm	Frequency modulation

[1] In some cases the subscript "max" is used to indicate maximum or peak current or voltage. For example, when the subscript "min" is used to indicate minimum current or voltage, the subscript "max" will probably be used for maximum current or voltage.

[2] In some cases the subscript "avg" is used to express average current or voltage.

[3] The abbreviations as given here are used as nouns. When they are used as adjectives, it is common practice to use a hyphen, thus

ac Alternating current (noun)
a-c alternating-current (adjective)

pm	Phase modulation
ptm	Pulse time modulation
emf	Electromotive force in volts
mmf	Magnetomotive force in gilberts
avc	Automatic volume control
davc	Delayed automatic volume control
mopa	Master-oscillator power-amplifier

f. **Measurement abbreviations:**

Metric prefixes

μ	Micro- (one-millionth)
m	Milli- (one-thousandth)
c	Centi- (one-hundredth)
d	Deci- (one-tenth)
	Uni- (one) NOTE: This prefix is usually omitted.
dk	Deka- (ten)
h	Hekto- (one hundred)
k	Kilo- (one thousand)
M	Mega- (one million)

Metric measure

mm	Millimeter (one-thousandth of a meter)
cm	Centimeter (one-hundredth of a meter)
km	Kilometer (one thousand meters)
sq cm	Square centimeter or cm^2
sq cm	Square meter or m^2
sq km	Square kilometer or km^3
cu cm	Cubic meter or m^3
cgs	Centimeter-gram-second system of units
mks	Meter-kilogram-second system of units

English measure

in.	Inch
ft	Foot
yd	Yard
mile	Mile NOTE: This unit is always spelled out.
sq in	Square inch
sq ft	Square foot
cu in	Cubic inch
cu ft	Cubic foot
sec	Second
min	Minute
hr	Hour
yr	Year
lb	Pound
rps	Revolutions per second
rpm	Revolutions per minute
mph	Miles per hour

g **Miscellaneous abbreviations:**

A.W.G. or B.&S. American wire gauge or Brown and Sharpe wire gauge. NOTE: These gauges are commonly used for copper, aluminum, and resistance wire.

db	Decibel
deg	Degree
diam	Diameter
dcc	Double cotton covered
dsc	Double silk covered
ec	Enamel covered
mlt	Mean length of turn
p., pp.	Page, pages
rms	Root mean square
rss	Root sum square
temp	Temperature
vol	Volume
vs.	Versus
wt	Weight

4. CONVERSION TABLE

Multiply number of by to obtain	$\mu\mu$ units micro-micro units	μ units micro units	m units milli units	units	k units kilo units	M units mega units
$\mu\mu$ units micromicro units	1	10^{6}	10^{9}	10^{12}	10^{15}	10^{18}
μ units micro units	10^{-6}	1	10^{3}	10^{6}	10^{9}	10^{12}
m units milli units	10^{-9}	10^{-3}	1	10^{3}	10^{6}	10^{9}
units	10^{-12}	10^{-6}	10^{-3}	1	10^{3}	10^{6}
k units kilo units	10^{-15}	10^{-9}	10^{-6}	10^{-3}	1	10^{3}
M units mega units	10^{-18}	10^{-12}	10^{-9}	10^{-6}	10^{-3}	1

5. LOGARITHMS OF NUMBERS FROM 1 TO 100

n	log n	n	log n	n	log n	n	log n	n	log n
1	0.000000	21	1.322219	41	1.612784	61	1.785330	81	1.908485
2	0.301030	22	1.342423	42	1.623249	62	1.792392	82	1.913814
3	0.477121	23	1.361728	43	1.633468	63	1.799341	83	1.919078
4	0.602060	24	1.380211	44	1.643453	64	1.806180	84	1.924279
5	0.698970	25	1.397940	45	1.653213	65	1.812913	85	1.929419
6	0.778151	26	1.414973	46	1.662758	66	1.819544	86	1.934498
7	0.845098	27	1.431364	47	1.672098	67	1.826075	87	1.939519
8	0.903090	28	1.447158	48	1.681241	68	1.832509	88	1.944483
9	0.954243	29	1.462398	49	1.690196	69	1.838849	89	1.949390
10	1.000000	30	1.477121	50	1.698970	70	1.845098	90	1.954243
11	1.041393	31	1.491362	51	1.707570	71	1.851258	91	1.959041
12	1.079181	32	1.505150	52	1.716003	72	1.857332	92	1.963788
13	1.113943	33	1.518514	53	1.724276	73	1.863323	93	1.968483
14	1.146128	34	1.531479	54	1.732394	74	1.869232	94	1.973128
15	1.176091	35	1.544068	55	1.740363	75	1.875061	95	1.977724
16	1.204120	36	1.556303	56	1.748188	76	1.880814	96	1.982271
17	1.230449	37	1.568202	57	1.755875	77	1.886491	97	1.986772
18	1.255273	38	1.579784	58	1.763428	78	1.892095	98	1.991226
19	1.278754	39	1.591065	59	1.770852	79	1.897627	99	1.995635
20	1.301030	40	1.602060	60	1.778151	80	1.903090	100	2.000000

6. BRIEF TABLE OF TRIGONOMETRIC FUNCTIONS

Degrees	Sine	Cosine	Tangent	Cotangent
0	0.0000	1.0000	0.0000	∞
10	0.1736	0.9848	0.1763	5.6713
20	0.3420	0.9397	0.3640	2.7475
30	0.5000	0.8660	0.5774	1.7321
40	0.6428	0.7660	0.8391	1.1918
50	0.7660	0.6428	1.1918	0.8391
60	0.8660	0.5000	1.7321	0.5774
70	0.9397	0.3420	2.7475	0.3640
80	0.9848	0.1736	5.6713	0.1763
90	1.0000	0.0000	∞	0.0000

7. TABLE OF NATURAL LOGARITHMS*

These four pages give the natural logarithms of numbers between 0.1 and 10, correct to four places. Moving the decimal point n places to the right in the number, *i.e.*, multiplying the number by 10^n, requires the addition of n times 2.3026 to the logarithm; values are given in the accompanying tables. Since the numbers in the first two pages are less than 1, the logarithms are all *negative* and should be preceded by a minus sign. The base is $e = 2.71828$.

$$\log_e x = \log_e 10 \log_{10} x = 2.3026 \log_{10} x$$
$$\log_{10} x = \log_{10} e \log_e x = 0.4343 \log_e x$$

n	n(2.3026)	n(0.6974−3)
1	2.3026	0.6974−3
2	4.6052	0.3948−5
3	6.9078	0.0922−7
4	9.2103	0.7897−10
5	11.5129	0.4871−12
6	13.8155	0.1845−14
7	16.1181	0.8819−17
8	18.4207	0.5793−19
9	20.7233	0.2767−21

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
.10	2.3026	2926	2828	2730	2634	2538	2443	2349	2256	2164	95
.11	.2073	1982	1893	1804	1716	1628	1542	1456	1371	1286	87
.12	.1203	1120	1037	0956	0875	0794	0715	0636	0557	0479	80
.13	.0402	0326	0250	0174	0099	0025	9951	9878	9805	9733	74
.14	1.9661	9590	9519	9449	9379	9310	9241	9173	9105	9038	69
.15	1.8971	8905	8839	8773	8708	8643	8579	8515	8452	8389	65
.16	8326	8264	8202	8140	8079	8018	7958	7898	7838	7779	61
.17	7720	7661	7603	7545	7487	7430	7373	7316	7260	7204	57
.18	7148	7093	7037	6983	6928	6874	6820	6766	6713	6660	54
.19	6607	6555	6503	6451	6399	6348	6296	6246	6195	6145	51
.20	1.6094	6045	5995	5945	5896	5847	5799	5750	5702	5654	49
.21	5606	5559	5512	5465	5418	5371	5325	5279	5233	5187	47
.22	5141	5096	5051	5006	4961	4917	4872	4828	4784	4740	44
.23	4697	4653	4610	4567	4524	4482	4439	4397	4355	4313	43
.24	4271	4230	4188	4147	4106	4065	4024	3984	3943	3903	41
.25	1.3863	3823	3783	3744	3704	3665	3626	3587	3548	3509	39
.26	3471	3432	3394	3356	3318	3280	3243	3205	3168	3130	38
.27	3093	3056	3020	2983	2946	2910	2874	2837	2801	2765	36
.28	2730	2694	2658	2623	2588	2553	2518	2483	2448	2413	35
.29	2379	2344	2310	2276	2242	2208	2174	2140	2107	2073	34
.30	1.2040	2006	1973	1940	1907	1874	1842	1809	1777	1744	33
.31	1712	1680	1648	1616	1584	1552	1520	1489	1457	1426	32
.32	1394	1363	1332	1301	1270	1239	1209	1178	1147	1117	31
.33	1087	1056	1026	0996	0966	0936	0906	0877	0847	0818	30
.34	0788	0759	0729	0700	0671	0642	0613	0584	0556	0527	29
.35	1.0498	0470	0441	0413	0385	0356	0328	0300	0272	0244	28
.36	0217	0189	0161	0134	0106	0079	0051	0024	9997	9970	27
.37	0.9943	9916	9889	9862	9835	9808	9782	9755	9729	9702	27
.38	9676	9650	9623	9597	9571	9545	9519	9493	9467	9442	26
.39	9416	9390	9365	9339	9314	9289	9263	9238	9213	9188	25
.40	0.9163	9138	9113	9088	9063	9039	9014	8989	8965	8940	25
.41	8916	8892	8867	8843	8819	8795	8771	8747	8723	8699	24
.42	8675	8651	8627	8604	8580	8557	8533	8510	8486	8463	23
.43	8440	8416	8393	8370	8347	8324	8301	8278	8255	8233	23
.44	8210	8187	8164	8142	8119	8097	8074	8052	8030	8007	22
.45	0.7985	7963	7941	7919	7897	7875	7853	7831	7809	7787	22
.46	7765	7744	7722	7700	7679	7657	7636	7614	7593	7572	21
.47	7550	7529	7508	7487	7465	7444	7423	7402	7381	7361	21
.48	7340	7319	7298	7277	7257	7236	7215	7195	7174	7154	20
.49	7133	7113	7093	7072	7052	7032	7012	6992	6972	6951	20

* This page and the following are from Allen's "Six-Place Tables," published by the McGraw-Hill Book Company, Inc., New York.

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
.50	0.6931	6911	6892	6872	6852	6832	6812	6792	6773	6753	20
.51	6733	6714	6694	6675	6655	6636	6616	6597	6578	6559	19
.52	6539	6520	6501	6482	6463	6444	6425	6406	6387	6368	19
.53	6349	6330	6311	6292	6274	6255	6236	6218	6199	6180	18
.54	6162	6143	6125	6106	6088	6070	6051	6033	6015	5997	18
.55	0.5978	5960	5942	5924	5906	5888	5870	5852	5834	5816	18
.56	5798	5780	5763	5745	5727	5709	5692	5674	5656	5639	18
.57	5621	5604	5586	5569	5551	5534	5516	5499	5482	5465	17
.58	5447	5430	5413	5396	5379	5361	5344	5327	5310	5293	17
.59	5276	5259	5243	5226	5209	5192	5175	5158	5142	5125	17
.60	0.5108	5092	5075	5058	5042	5025	5009	4992	4976	4959	16
.61	4943	4927	4910	4894	4878	4861	4845	4829	4813	4796	16
.62	4780	4764	4748	4732	4716	4700	4684	4668	4652	4636	16
.63	4620	4604	4589	4573	4557	4541	4526	4510	4494	4479	16
.64	4463	4447	4432	4416	4401	4385	4370	4354	4339	4323	15
.65	0.4308	4292	4277	4262	4246	4231	4216	4201	4186	4170	15
.66	4155	4140	4125	4110	4095	4080	4065	4050	4035	4020	15
.67	4005	3990	3975	3960	3945	3930	3916	3901	3886	3871	15
.68	3857	3842	3827	3813	3798	3783	3769	3754	3740	3725	15
.69	3711	3696	3682	3667	3653	3638	3624	3610	3595	3581	14
.70	0.3567	3552	3538	3524	3510	3496	3481	3467	3453	3439	14
.71	3425	3411	3397	3383	3369	3355	3341	3327	3313	3299	14
.72	3285	3271	3257	3243	3230	3216	3202	3188	3175	3161	14
.73	3147	3133	3120	3106	3092	3079	3065	3052	3038	3025	13
.74	3011	2998	2984	2971	2957	2944	2930	2917	2904	2890	13
.75	0.2877	2863	2850	2837	2824	2810	2797	2784	2771	2758	13
.76	2744	2731	2718	2705	2692	2679	2666	2653	2640	2627	13
.77	2614	2601	2588	2575	2562	2549	2536	2523	2510	2497	13
.78	2485	2472	2459	2446	2433	2421	2408	2395	2383	2370	13
.79	2357	2345	2332	2319	2307	2294	2282	2269	2256	2244	12
.80	0.2231	2219	2206	2194	2182	2169	2157	2144	2132	2120	12
.81	2107	2095	2083	2070	2058	2046	2033	2021	2009	1997	12
.82	1985	1972	1960	1948	1936	1924	1912	1900	1888	1875	12
.83	1863	1851	1839	1827	1815	1803	1791	1779	1767	1755	12
.84	1744	1732	1720	1708	1696	1684	1672	1661	1649	1637	12
.85	0.1625	1613	1602	1590	1578	1567	1555	1543	1532	1520	12
.86	1508	1497	1485	1473	1462	1450	1439	1427	1416	1404	12
.87	1393	1381	1370	1358	1347	1335	1324	1312	1301	1290	11
.88	1278	1267	1256	1244	1233	1222	1210	1199	1188	1177	11
.89	1166	1154	1143	1132	1120	1109	1098	1087	1076	1065	11
.90	0.1054	1043	1031	1020	1009	0998	0987	0976	0965	0954	11
.91	0943	0932	0921	0910	0899	0888	0877	0866	0856	0845	11
.92	0834	0823	0812	0801	0790	0780	0769	0758	0747	0736	11
.93	0726	0715	0704	0693	0683	0672	0661	0651	0640	0629	11
.94	0619	0608	0598	0587	0576	0566	0555	0545	0534	0523	11
.95	0.0513	0502	0492	0481	0471	0460	0450	0440	0429	0419	10
.96	0408	0398	0387	0377	0367	0356	0346	0336	0325	0315	10
.97	0305	0294	0284	0274	0263	0253	0243	0233	0222	0212	10
.98	0202	0192	0182	0171	0161	0151	0141	0131	0121	0111	10
.99	0101	0090	0080	0070	0060	0050	0040	0030	0020	0010	10

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
1.0	0.0000	0100	0198	0296	0392	0488	0583	0677	0770	0862	95
1.1	0953	1044	1133	1222	1310	1398	1484	1570	1655	1740	87
1.2	1823	1906	1989	2070	2151	2231	2311	2390	2469	2546	80
1.3	2624	2700	2776	2852	2927	3001	3075	3148	3221	3293	74
1.4	3365	3436	3507	3577	3646	3716	3784	3853	3920	3988	69
1.5	0.4055	4121	4187	4253	4318	4383	4447	4511	4574	4637	65
1.6	4700	4762	4824	4886	4947	5008	5068	5128	5188	5247	61
1.7	5306	5365	5423	5481	5539	5596	5653	5710	5766	5822	57
1.8	5878	5933	5988	6043	6098	6152	6206	6259	6313	6366	54
1.9	6419	6471	6523	6575	6627	6678	6729	6780	6831	6881	51
2.0	0.6931	6981	7031	7080	7129	7178	7227	7275	7324	7372	49
2.1	7419	7467	7514	7561	7608	7655	7701	7747	7793	7839	47
2.2	7885	7930	7975	8020	8065	8109	8154	8198	8242	8286	44
2.3	8329	8372	8416	8459	8502	8544	8587	8629	8671	8713	43
2.4	8755	8796	8838	8879	8920	8961	9002	9042	9083	9123	41
2.5	0.9163	9203	9243	9282	9322	9361	9400	9439	9478	9517	39
2.6	9555	9594	9632	9670	9708	9746	9783	9821	9858	9895	38
2.7	0.9933	9969	0006	0043	0080	0116	0152	0188	0225	0260	36
2.8	1.0296	0332	0367	0403	0438	0473	0508	0543	0578	0613	35
2.9	0647	0682	0716	0750	0784	0818	0852	0886	0919	0953	34
3.0	1.0986	1019	1053	1086	1119	1151	1184	1217	1249	1282	33
3.1	1314	1346	1378	1410	1442	1474	1506	1537	1569	1600	32
3.2	1632	1663	1694	1725	1756	1787	1817	1848	1878	1909	31
3.3	1939	1969	2000	2030	2060	2090	2119	2149	2179	2208	30
3.4	2238	2267	2296	2326	2355	2384	2413	2442	2470	2499	29
3.5	1.2528	2556	2585	2613	2641	2669	2698	2726	2754	2782	28
3.6	2809	2837	2865	2892	2920	2947	2975	3002	3029	3056	27
3.7	3083	3110	3137	3164	3191	3218	3244	3271	3297	3324	27
3.8	3350	3376	3403	3429	3455	3481	3507	3533	3558	3584	26
3.9	3610	3635	3661	3686	3712	3737	3762	3788	3813	3838	25
4.0	1.3863	3888	3913	3938	3962	3987	4012	4036	4061	4085	25
4.1	4110	4134	4159	4183	4207	4231	4255	4279	4303	4327	24
4.2	4351	4375	4398	4422	4446	4469	4493	4516	4540	4563	23
4.3	4586	4609	4633	4656	4679	4702	4725	4748	4770	4793	23
4.4	4816	4839	4861	4884	4907	4929	4951	4974	4996	5019	22
4.5	1.5041	5063	5085	5107	5129	5151	5173	5195	5217	5239	22
4.6	5261	5282	5304	5326	5347	5369	5390	5412	5433	5454	21
4.7	5476	5497	5518	5539	5560	5581	5602	5623	5644	5665	21
4.8	5686	5707	5728	5748	5769	5790	5810	5831	5851	5872	20
4.9	5892	5913	5933	5953	5974	5994	6014	6034	6054	6074	20
5.0	1.6094	6114	6134	6154	6174	6194	6214	6233	6253	6273	20
5.1	6292	6312	6332	6351	6371	6390	6409	6429	6448	6467	19
5.2	6487	6506	6525	6544	6563	6582	6601	6620	6639	6658	19
5.3	6677	6696	6715	6734	6752	6771	6790	6808	6827	6845	18
5.4	6864	6882	6901	6919	6938	6956	6974	6993	7011	7029	18
5.5	1.7047	7066	7084	7102	7120	7138	7156	7174	7192	7210	18
5.6	7228	7246	7263	7281	7299	7317	7334	7352	7370	7387	18
5.7	7405	7422	7440	7457	7475	7492	7509	7527	7544	7561	17
5.8	7579	7596	7613	7630	7647	7664	7681	7699	7716	7733	17
5.9	7750	7766	7783	7800	7817	7834	7851	7867	7884	7901	17

[1] This and following page taken from Marks' "Mechanical Engineers' Handbook," published by the McGraw-Hill Book Company, Inc., New York.

Number	0	1	2	3	4	5	6	7	8	9	Avg. diff.
6.0	1.7918	7934	7951	7967	7984	8001	8017	8034	8050	8066	16
6.1	8083	8099	8116	8132	8148	8165	8181	8197	8213	8229	16
6.2	8245	8262	8278	8294	8310	8326	8342	8358	8374	8390	16
6.3	8405	8421	8437	8453	8469	8485	8500	8516	8532	8547	16
6.4	8563	8579	8594	8610	8625	8641	8656	8672	8687	8703	15
6.5	1.8718	8733	8749	8764	8779	8795	8810	8825	8840	8856	15
6.6	8871	8886	8901	8916	8931	8946	8961	8976	8991	9006	15
6.7	9021	9036	9051	9066	9081	9095	9110	9125	9140	9155	15
6.8	9169	9184	9199	9213	9228	9242	9257	9272	9286	9301	15
6.9	9315	9330	9344	9359	9373	9387	9402	9416	9430	9445	14
7.0	1.9459	9473	9488	9502	9516	9530	9544	9559	9573	9587	14
7.1	9601	9615	9629	9643	9657	9671	9685	9699	9713	9727	14
7.2	9741	9755	9769	9782	9796	9810	9824	9838	9851	9865	14
7.3	1.9879	9892	9906	9920	9933	9947	9961	9974	9988	0001	13
7.4	2.0015	0028	0042	0055	0069	0082	0096	0109	0122	0136	13
7.5	2.0149	0162	0176	0189	0202	0215	0229	0242	0255	0268	13
7.6	0281	0295	0308	0321	0334	0347	0360	0373	0386	0399	13
7.7	0412	0425	0438	0451	0464	0477	0490	0503	0516	0528	13
7.8	0541	0554	0567	0580	0592	0605	0618	0631	0643	0656	13
7.9	0669	0681	0694	0707	0719	0732	0744	0757	0769	0782	12
8.0	2.0794	0807	0819	0832	0844	0857	0869	0882	0894	0906	12
8.1	0919	0931	0943	0956	0968	0980	0992	1005	1017	1029	12
8.2	1041	1054	1066	1078	1090	1102	1114	1126	1138	1150	12
8.3	1163	1175	1187	1199	1211	1223	1235	1247	1258	1270	12
8.4	1282	1294	1306	1318	1330	1342	1353	1365	1377	1389	12
8.5	2.1401	1412	1424	1436	1448	1459	1471	1483	1494	1506	12
8.6	1518	1529	1541	1552	1564	1576	1587	1599	1610	1622	12
8.7	1633	1645	1656	1668	1679	1691	1702	1713	1725	1736	11
8.8	1748	1759	1770	1782	1793	1804	1815	1827	1838	1849	11
8.9	1861	1872	1883	1894	1905	1917	1928	1939	1950	1961	11
9.0	2.1972	1983	1994	2006	2017	2028	2039	2050	2061	2072	11
9.1	2083	2094	2105	2116	2127	2138	2148	2159	2170	2181	11
9.2	2192	2203	2214	2225	2235	2246	2257	2268	2279	2289	11
9.3	2300	2311	2322	2332	2343	2354	2364	2375	2386	2396	11
9.4	2407	2418	2428	2439	2450	2460	2471	2481	2492	2502	11
9.5	2.2513	2523	2534	2544	2555	2565	2576	2586	2597	2607	10
9.6	2618	2628	2638	2649	2659	2670	2680	2690	2701	2711	10
9.7	2721	2732	2742	2752	2762	2773	2783	2793	2803	2814	10
9.8	2824	2834	2844	2854	2865	2875	2885	2895	2905	2915	10
9.9	2925	2935	2946	2956	2966	2976	2986	2996	3006	3016	10
10.0	2.3026										

8. TABLE OF EXPONENTIAL AND HYPERBOLIC FUNCTIONS IN TERMS OF NEPERS*

x	e^x	e^{-x}	sinh x	cosh x	tanh x
.00	1.0000	1.0000	0.0000	1.0000	0.0000
.01	.0101	0.9900	.0100	.0001	.0100
.02	.0202	.9802	.0200	.0002	.0200
.03	.0305	.9704	.0300	.0005	.0300
.04	.0408	.9608	.0400	.0008	.0400
.05	.0513	.9512	.0500	.0013	.0500
.06	.0618	.9418	.0600	.0018	.0599
.07	.0725	.9324	.0701	.0025	.0699
.08	.0833	.9231	.0801	.0032	.0798
.09	.0942	.9139	.0901	.0041	.0898
.10	1.1052	0.9048	0.1002	1.0050	0.0997
.11	.1163	.8958	.1102	.0061	.1096
.12	.1275	.8869	.1203	.0072	.1194
.13	.1388	.8781	.1304	.0085	.1293
.14	.1503	.8694	.1405	.0098	.1391
.15	.1618	.8607	.1506	.0113	.1489
.16	.1735	.8521	.1607	.0128	.1587
.17	.1853	.8437	.1708	.0145	.1684
.18	.1972	.8353	.1810	.0162	.1781
.19	.2092	.8270	.1911	.0181	.1878
.20	1.2214	0.8187	0.2013	1.0201	0.1974
.21	.2337	.8106	.2115	.0221	.2070
.22	.2461	.8025	.2218	.0243	.2165
.23	.2586	.7945	.2320	.0266	.2260
.24	.2712	.7866	.2423	.0289	.2355
.25	.2840	.7788	.2526	.0314	.2449
.26	.2969	.7711	.2629	.0340	.2543
.27	.3100	.7634	.2733	.0367	.2636
.28	.3231	.7558	.2837	.0395	.2729
.29	.3364	.7483	.2941	.0423	.2821
.30	1.3499	0.7408	0.3045	1.0453	0.2913
.31	.3634	.7334	.3150	.0484	.3004
.32	.3771	.7261	.3255	.0516	.3095
.33	.3910	.7189	.3360	.0549	.3185
.34	.4049	.7118	.3466	.0584	.3275
.35	.4191	.7047	.3572	.0619	.3364
.36	.4333	.6977	.3678	.0655	.3452
.37	.4477	.6907	.3785	.0692	.3540
.38	.4623	.6839	.3892	.0731	.3627
.39	.4770	.6771	.4000	.0770	.3714
.40	1.4918	0.6703	0.4108	1.0811	0.3800
.41	.5068	.6637	.4216	.0852	.3885
.42	.5220	.6570	.4325	.0895	.3969
.43	.5373	.6505	.4434	.0939	.4053
.44	.5527	.6440	.4543	.0984	.4136
.45	.5683	.6376	.4653	.1030	.4219
.46	.5841	.6313	.4764	.1077	.4301
.47	.6000	.6250	.4875	.1125	.4382
.48	.6161	.6188	.4986	.1174	.4462
.49	.6323	.6126	.5098	.1225	.4542
.50	1.6487	0.6065	0.5211	1.1276	0.4621
.51	.6653	.6005	.5324	.1329	.4700
.52	.6820	.5945	.5438	.1383	.4777
.53	.6989	.5886	.5552	.1438	.4854
.54	.7160	.5827	.5666	.1494	.4930

* From Allen's "Six-Place Tables," published by the McGraw-Hill Book Company, Inc., New York.

x	e^x	e^{-x}	sinh x	cosh x	tanh x
.55	.7333	.5769	.5782	.1551	.5005
.56	.7507	.5712	.5897	.1609	.5080
.57	.7683	.5655	.6014	.1669	.5154
.58	.7860	.5599	.6131	.1730	.5227
.59	.8040	.5543	.6248	.1792	.5299
.60	1.8221	0.5488	0.6367	1.1855	0.5370
.61	.8404	.5434	.6485	.1919	.5441
.62	.8589	.5379	.6605	.1984	.5511
.63	.8776	.5326	.6725	.2051	.5581
.64	.8965	.5273	.6846	.2119	.5649
.65	.9155	.5220	.6967	.2188	.5717
.66	.9348	.5169	.7090	.2258	.5784
.67	.9542	.5117	.7213	.2330	.5850
.68	.9739	.5066	.7336	.2402	.5915
.69	.9937	.5016	.7461	.2476	.5980
.70	2.0138	0.4966	0.7586	1.2552	0.6044
.71	.0340	.4916	.7712	.2628	.6107
.72	.0544	.4868	.7838	.2706	.6169
.73	.0751	.4819	.7966	.2785	.6231
.74	.0959	.4771	.8094	.2865	.6291
.75	.1170	.4724	.8223	.2947	.6352
.76	.1383	.4677	.8353	.3030	.6411
.77	.1598	.4630	.8484	.3114	.6469
.78	.1815	.4584	.8615	.3199	.6527
.79	.2034	.4538	.8748	.3286	.6584
.80	2.2255	0.4493	0.8881	1.3374	0.6640
.81	.2479	.4449	.9015	.3464	.6696
.82	.2705	.4404	.9150	.3555	.6751
.83	.2933	.4360	.9286	.3647	.6805
.84	.3164	.4317	.9423	.3740	.6858
.85	.3396	.4274	.9561	.3835	.6911
.86	.3632	.4232	.9700	.3932	.6963
.87	.3869	.4190	.9840	.4029	.7014
.88	.4109	.4148	.9981	.4128	.7064
.89	.4351	.4107	1.0122	.4229	.7114
.90	2.4596	0.4066	1.0265	1.4331	0.7163
.91	.4843	.4025	.0409	.4434	.7211
.92	.5093	.3985	.0554	.4539	.7259
.93	.5345	.3946	.0700	.4645	.7306
.94	.5600	.3906	.0847	.4753	.7352
.95	.5857	.3867	.0995	.4862	.7398
.96	.6117	.3829	.1144	.4973	.7443
.97	.6379	.3791	.1294	.5085	.7487
.98	.6645	.3753	.1446	.5199	.7531
.99	.6912	.3716	.1598	.5314	.7574
1.00	2.7183	0.3679	1.1752	1.5431	0.7616
.05	.8577	.3499	.2539	.6038	.7818
.10	3.0042	.3329	.3356	.6685	.8005
.15	.1582	.3166	.4208	.7374	.8178
.20	.3201	.3012	.5095	.8107	.8337
.25	.4903	.2865	.6019	.8884	.8483
.30	.6693	.2725	.6984	.9709	.8617
.35	.8574	.2592	.7991	2.0583	.8741
.40	4.0552	.2466	.9043	.1509	.8854
.45	.2631	.2346	2.0143	.2488	.8957

x	e^x	e^{-x}	sinh x	cosh x	tanh x
1.50	4.4817	0.2231	2.1293	2.3524	0.9052
.55	.7115	.2122	.2496	.4619	.9138
.60	.9530	.2019	.3756	.5775	.9217
.65	5.2070	.1920	.5075	.6995	.9289
.70	.4739	.1827	.6456	.8283	.9354
.75	.7546	.1738	.7904	.9642	.9414
.80	6.0496	.1653	.9422	3.1075	.9468
.85	.3598	.1572	3.1013	.2585	.9518
.90	.6859	.1496	.2682	.4177	.9562
.95	7.0287	.1423	.4432	.5855	.9603
2.00	7.3891	0.1353	3.6269	3.7622	0.9640
.05	.7679	.1287	.8196	.9483	.9674
.10	8.1662	.1225	4.0219	4.1443	.9705
.15	.5848	.1165	.2342	.3507	.9732
.20	9.025	.1108	.457	.568	.9757
.25	.488	.1054	.691	.797	.9780
.30	.974	.1003	.937	5.037	.9801
.35	10.486	.0954	5.195	.290	.9820
.40	11.023	.0907	.466	.557	.9837
.5	12.182	.0821	6.050	6.132	.9866
.6	13.464	.0743	.695	.769	.9890
.7	14.880	.0672	7.406	7.473	.9910
.8	16.445	.0608	8.192	8.253	.9926
.9	18.174	.0550	9.060	9.115	.9940
3.0	20.086	.0498	10.018	10.068	.9951
.1	22.198	.0450	11.076	11.121	.9959
.2	24.533	.0408	12.246	12.287	.9967
.3	27.113	.0369	13.538	13.575	.9973
.4	29.964	.0334	14.965	14.999	.9978
.5	33.115	.0302	16.543	16.573	.9982
.6	36.598	.0273	18.285	18.313	.9985
.7	40.447	.0247	20.211	20.236	.9988
.8	44.701	.0224	22.339	22.362	.9990
.9	49.402	.0202	24.691	24.711	.9992
4.0	54.60	.0183	27.29	27.31	.999
4.1	60.34	.0166	30.16	30.18	.999
4.2	66.69	.0150	33.34	33.35	1.000
4.3	73.70	.0136	36.84	36.86	1.000
4.4	81.45	.0123	40.72	40.73	1.000
4.5	90.02	.0111	45.00	45.01	1.000
4.6	99.48	.0101	49.74	49.75	1.000
4.7	109.95	.0091	54.97	54.98	1.000
4.8	121.51	.0082	60.75	60.76	1.000
4.9	134.29	.0074	67.14	67.15	1.000
5.0	148.41	.0067	74.20	74.21	1.000
6.0	403.4	.0025	201.7		1.000
7.0	1096.6	.00091	548.3		1.000
8.0	2981.0	.00034	1490.5		1.000
9.0	8103.1	.00012	4051.5		1.000
10.0	22026.5	.000045	11013.2		1.000

In the compilation of this table, extensive use was made of Kennelly's Tables of Complex Hyperbolic and Circular Functions.

9. TABLE OF EXPONENTIAL AND HYPERBOLIC FUNCTIONS IN TERMS OF DECIBELS

N_{db}	Nepers α	k ϵ^{α}	$\frac{1}{k}$ $\epsilon^{-\alpha}$	k^2 $\epsilon^{2\alpha}$	$\frac{k^2-1}{2k}$ $\sinh \alpha$	$\frac{k^2+1}{2k}$ $\cosh \alpha$	$\frac{k^2-1}{k^2+1}$ $\tanh \alpha$	$\frac{k-1}{k+1}$ $\tanh \frac{\alpha}{2}$
0.05	0.005757	1.00577	0.99426	1.01158	0.005756	1.00002	0.005756	0.002878
0.1	0.011513	1.01158	0.98855	1.02330	0.011513	1.00007	0.011512	0.005756
0.2	0.023024	1.02329	0.97724	1.04713	0.023028	1.00027	0.023022	0.011512
0.3	0.034539	1.03514	0.96605	1.07152	0.034546	1.00060	0.034525	0.017268
0.4	0.046052	1.04713	0.95499	1.09648	0.046067	1.00106	0.046019	0.023022
0.5	0.057565	1.05925	0.94406	1.12202	0.057597	1.00166	0.057502	0.028774
0.6	0.069078	1.07152	0.93325	1.14815	0.069131	1.00238	0.068968	0.034525
0.7	0.080591	1.08393	0.92257	1.17490	0.080679	1.00325	0.080418	0.040274
0.8	0.092103	1.09648	0.91201	1.20227	0.092236	1.00425	0.091846	0.046019
0.9	0.103626	1.10917	0.90157	1.23027	0.103797	1.00538	0.103249	0.051763
1.0	0.11513	1.12202	0.89125	1.25893	0.11539	1.00664	0.11463	0.057502
1.1	0.12664	1.13501	0.88105	1.28825	0.12697	1.00803	0.12597	0.063237
1.2	0.13816	1.14815	0.87096	1.31826	0.13860	1.00956	0.13728	0.068968
1.3	0.14967	1.16145	0.86099	1.34896	0.15023	1.01122	0.14956	0.074695
1.4	0.16118	1.17490	0.85114	1.38038	0.16188	1.01301	0.15979	0.080418
1.5	0.17269	1.18850	0.84139	1.41254	0.17356	1.01495	0.17100	0.086132
1.6	0.18421	1.20227	0.83176	1.44544	0.18525	1.01701	0.18215	0.091846
1.7	0.19572	1.21618	0.82224	1.47911	0.19697	1.01922	0.19326	0.097551
1.8	0.20723	1.23027	0.81283	1.51356	0.20872	1.02155	0.20432	0.103249
1.9	0.21876	1.24452	0.80353	1.54882	0.22050	1.02402	0.21532	0.108939
2.0	0.23026	1.25893	0.79433	1.58489	0.23230	1.02662	0.22627	0.11463
2.2	0.25328	1.28825	0.77625	1.65959	0.25600	1.03225	0.24800	0.12597
2.4	0.27631	1.31826	0 75858	1.73780	0.27984	1.03841	0.26948	0.13728
2.5	0.28782	1.33352	0.74989	1.77828	0.29181	1.04171	0.28013	0.14293
2.6	0.29934	1.34896	0.74131	1.81970	0.30383	1.04514	0.29071	0.14856
2.8	0.32236	1.38038	0.72444	1.90546	0.32798	1.05241	0.31164	0.15980
3.0	0.34539	1.41254	0.70795	1.99526	0.35230	1.06024	0.33228	0.17100
3.2	0.36841	1.44544	0.69183	2.08930	0.37681	1.06864	0.35261	0.18215
3.4	0.39144	1.47911	0.67608	2.18776	0.40151	1.07759	0.37260	0.19326
3.5	0.40295	1.49623	0.66834	2.23870	0.41394	1.08229	0.38246	0.19879
3.6	0.41447	1.51356	0.66069	2.29086	0.42645	1.08714	0.39226	0.20432
3.8	0.43749	1.54882	0.64565	2.39884	0.45157	1.09722	0.41156	0.21532
4.0	0.46052	1.58489	0.63096	2.51188	0.47697	1.10793	0.43051	0.22627
4.5	0.51808	1.67880	0.59566	2.81837	0.54158	1.13724	0.47622	0.25340
5.0	0.57565	1.77828	0.56234	3.16228	0.60798	1.17032	0.51950	0.28013

N_{db}	Nepers α	k ϵ^{α}	$\frac{1}{k}$ $\epsilon^{-\alpha}$	k^2 $\epsilon^{2\alpha}$	$\frac{k^2-1}{2k}$ sinh α	$\frac{k^2+1}{2k}$ cosh α	$\frac{k^2-1}{k^2+1}$ tanh α	$\frac{k-1}{k+1}$ tanh $\frac{\alpha}{2}$
5.5	0.63321	1.88365	0.53088	3.5481	0.67637	1.20726	0.56026	0.30643
6.0	0.69078	1.99526	0.50119	3.9811	0.74705	1.24823	0.59848	0.33228
6.5	0.74834	2.11349	0.47315	4.4668	0.82016	1.29331	0.63415	0.35764
7.0	0.80591	2.23872	0.44668	5.0119	0.89603	1.34272	0.66732	0.38246
7.5	0.86347	2.37137	0.42170	5.6234	0.97483	1.39652	0.69804	0.40677
8.0	0.92103	2.5119	0.39811	6.3096	1.05689	1.45500	0.72639	0.43051
8.5	0.97860	2.6607	0.37584	7.0795	1.14246	1.51830	0.75246	0.45366
9.0	1.03616	2.8184	0.35481	7.9433	1.23178	1.58659	0.77637	0.47622
9.5	1.09373	2.9854	0.33497	8.9125	1.32520	1.66016	0.79823	0.49817
10.0	1.15129	3.1623	0.31623	10.0000	1.42302	1.73924	0.81818	0.51950
10.5	1.20886	3.3497	0.29854	11.220	1.52551	1.82404	0.83633	0.54020
11.0	1.26642	3.5481	0.28184	12.589	1.63313	1.91497	0.85282	0.56026
11.5	1.32399	3.7584	0.26607	14.125	1.74609	2.01216	0.86777	0.57969
12.0	1.38155	3.9811	0.25119	15.849	1.86494	2.11612	0.88130	0.59848
12.5	1.43912	4.2170	0.23714	17.783	1.98992	2.22706	0.89352	0.61664
13.0	1.49668	4.4668	0.22387	19.953	2.12154	2.34542	0.90455	0.63416
13.5	1.55425	4.7315	0.21135	22.387	2.26007	2.47142	0.91448	0.65105
14.0	1.61181	5.0119	0.19953	25.119	2.40617	2.60570	0.92343	0.66733
14.5	1.66937	5.3088	0.18836	23.184	2.56028	2.74864	0.93147	0.68298
15.0	1.72694	5.6234	0.17783	31.623	2.72282	2.90065	0.93869	0.69804
15.5	1.78450	5.9566	0.16788	35.481	2.89435	3.06223	0.94517	0.71250
16.0	1.84207	6.3096	0.15849	39.811	3.07555	3.23404	0.95099	0.72639
16.5	1.89963	6.6834	0.14962	44.668	3.26690	3.41653	0.95621	0.73970
17.0	1.95720	7.0795	0.14125	50.119	3.46910	3.61035	0.96088	0.75246
17.5	2.01476	7.4989	0.13335	56.234	3.68281	3.81616	0.96506	0.76468
18.0	2.07233	7.9433	0.12589	63.096	3.90870	4.03460	0.96880	0.77637
18.5	2.12989	8.4139	0.11885	70.795	4.14760	4.26645	0.97214	0.78755
19.0	2.18746	8.9125	0.11220	79.433	4.40017	4.51237	0.97514	0.79823
19.5	2.24502	9.4406	0.10593	89.125	4.66734	4.77327	0.97781	0.80844
20.0	2.30259	10.0000	0.10000	100.000	4.95000	5.05000	0.98020	0.81818
21.0	2.41771	11.2202	8912.5*	125.893	5.56554	5.65467	0.98424	0.83634
22.0	2.53284	12.5893	7943.3*	158.490	6.25506	6.33450	0.98746	0.85282
23.0	2.64797	14.1254	7079.5*	199.527	7.02761	7.09841	0.99003	0.86777
24.0	2.76310	15.8489	6309.6*	251.188	7.89293	7.95602	0.99207	0.88130
25.0	2.87823	17.7828	5623.4*	316.228	8.86324	8.91947	0.99370	0.89352

* Multiply by 10^{-4}.

N_{db}	Nepers	k	$\frac{1}{k}$	k^2	$\frac{k^2-1}{2k}$	$\frac{k^2+1}{2k}$	$\frac{k^2-1}{k^2+1}$	$\frac{k-1}{k+1}$
	α	ϵ^{α}	$\epsilon^{-\alpha}$	$\epsilon^{2\alpha}$	sinh α	cosh α	tanh α	tanh $\frac{\alpha}{2}$
26.0	2.99336	19.953	5011.9*	398.11	9.951	10.001	0.99499	0.90455
27.0	3.10849	22.387	4466.8*	501.19	11.171	11.216	0.99602	0.91448
28.0	3.22362	25.119	3981.1*	630.96	12.540	12.579	0.99684	0.92343
29.0	3.33875	28.184	3548.1*	794.33	14.074	14.110	0.99749	0.93147
30.0	3.45388	31.623	3162.3*	1000.00	15.795	15.827	0.99800	0.93869
31.0	3.56901	35.481	2818.4*	1258.9	17.726	17.754	0.99841	0.94518
32.0	3.68414	39.811	2511.9*	1584.9	19.893	19.918	0.99874	0.95099
33.0	3.79927	44.668	2238.7*	1995.3	22.324	22.346	0.99900	0.95621
34.0	3.91439	50.119	1995.3*	2511.9	25.049	25.069	0.99920	0.96088
35.0	4.02952	56.234	1778.3*	3162.3	28.108	28.126	0.99937	0.96506
36.0	4.14465	63.096	1584.9*	3981.1	31.540	31.556	0.99950	0.96880
37.0	4.25978	70.795	1412.5*	5011.9	35.390	35.404	0.99960	0.97214
38.0	4.37491	79.433	1258.9*	6309.6	39.710	39.723	0.99968	0.97513
39.0	4.49004	89.125	1122.0*	7943.3	44.557	44.568	0.99975	0.97781
40.0	4.60517	100.000	1000.0*	10000.0	49.995	50.005	0.99980	0.98020
41.0	4.72030	112.20	891.25*	12589.	56.095	56.104	0.99984	0.98233
42.0	4.83543	125.89	794.33*	15849.	62.944	62.956	0.99987	0.98424
43.0	4.95056	141.25	707.95*	19953.	70.627	70.634	0.99990	0.98594
44.0	5.06569	158.49	630.96*	25119.	79.242	79.248	0.99992	0.98746
45.0	5.18081	177.83	562.34*	31623.	88.911	88.916	0.99994	0.98887
46.0	5.29595	199.53	501.19*	39811.	99.77	99.76	0.99995	0.99003
47.0	5.41107	223.87	446.68*	50119.	111.94	111.94	0.99996	0.99111
48.0	5.52620	251.19	398.11*	63096.	125.59	125.60	0.99997	0.99207
49.0	5.64133	281.84	354.81*	79433.	140.92	140.92	0.99997	0.99293
50.0	5.75646	316.23	316.23*	100000.	158.11	158.11	0.99998	0.99370
55.0	6.33211	562.3	177.83*	3.162†	281.17	281.17	0.99999	0.99645
60.0	6.90776	1000.0	100.00*	10.000†	500.00	500.00	0.99999	0.99800
65.0	7.48340	1778.3	56.23*	31.623†	889.14	889.14	0.99999	0.99888
70.0	8.05905	3162.3	31.62*	100.000†	1581.13	1581.13	1.00000	0.99937
75.0	8.63469	5623.4	17.78*	316.228†	2811.71	2811.71	1.00000	0.99964
80.0	9.21034	10000.	10.000*	1000.†	0.5000†	0.5000†	1.00000	0.99980
85.0	9.78599	17783.	5.623*	3162.†	0.8891†	0.8891†	1.00000	0.99989
90.0	10.36163	31623.	3.162*	10000.†	1.5811†	1.5811†	1.00000	0.99994
95.0	10.93728	56234.	1.778*	31623.†	2.8117†	2.8117†	1.00000	0.99996
100.0	11.51293	100000.	1.000*	100000.†	5.0000†	5.0000†	1.00000	0.99998

* Multiply by 10^{-5}.
† Multiply by 10^{+5}.

10. ELECTRON TUBE SYMBOLS

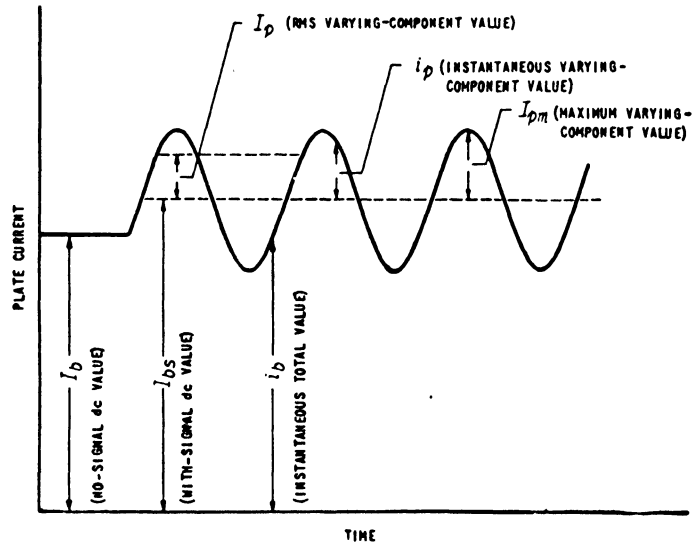
a. Letter symbols:

j	General (convention for any electrode)
f	Filament
h	Heater
k	Cathode
g	Grid
p	Plate or anode
s	Metal shell, or other self-shielding envelope
d	Deflecting, reflecting or repelling electrode (electrostatic type)
e_c	Instantaneous total grid voltage
e_b	Instantaneous total plate voltage
i_c	Instantaneous total grid current
i_b	Instantaneous total plate current
E_c	No signal d-c value of grid voltage
E_b	No signal d-c value of plate voltage
I_c	No signal d-c value of grid current
I_b	No signal d-c value of plate current
E_{cs}	With signal d-c value of grid voltage
E_{bs}	With signal d-c value of plate voltage
I_{cs}	With signal d-c value of grid current
I_{bs}	With signal d-c value of plate current
e_g	Instantaneous value of varying component of grid voltage
e_p	Instantaneous value of varying component of plate voltage
E_g	Effective value of varying component of grid voltage
E_p	Effective value of varying component of plate voltage
I_g	Effective value of varying component of grid current
I_p	Effective value of varying component of plate current
E_{gm}	Maximum value of varying component of grid voltage
E_{pm}	Maximum value of varying component of plate voltage
I_{gm}	Maximum value of varying component of grid voltage
I_{pm}	Maximum value of varying component of plate voltage
E_f	Filament or heater terminal voltage (effective value)
I_f	Filament or heater current (effective value)
I_s	Total electron emission
g_j	Conductance of electrode j
r_j	Resistance of electrode j
g_p	Plate conductance
r_p	Plate resistance
g_g	Grid conductance
r_g	Grid resistance
g_{jk}	Transconductance from electrode k to electrode j
g_m	($\equiv g_{pg}$) grid-plate transconductance (mutual conductance), effect in plate circuit to change on control grid.
g_n	($\equiv g_{pg}$) plate-grid transconductance (inverse mutual conductance), effect in grid circuit to change on plate
μ	Amplification factor

$\mu_{j_3 j_2}$	μ factor of an electron tube, relative effect of change on electrode "$j3$" to change on electrode "$j2$" (condition of other electrodes to be specified)
C_{gp}	Grid-plate capacitance
C_{gk}	Grid-cathode capacitance
C_{pk}	Plate-cathode capacitance
C_{gh}	Grid-heater capacitance
C_{ph}	Plate-heater capacitance
C_g	Grid capacitance
C_p	Plate capacitance
C_k	Cathode capacitance
E_{inv}	Peak (or crest) inverse voltage
E_{fwd}	Peak (or crest) forward voltage
t_d	Deionization time
t_h	Tube heating time
t_k	Cathode heating time
P_g	Grid dissipation power
P_p	Anode dissipation power
P_i	Input power
P_o	Output power
E_{cc}	Grid supply voltage
E_{bb}	Plate supply voltage
E_{ff}	Filament or heater supply voltage

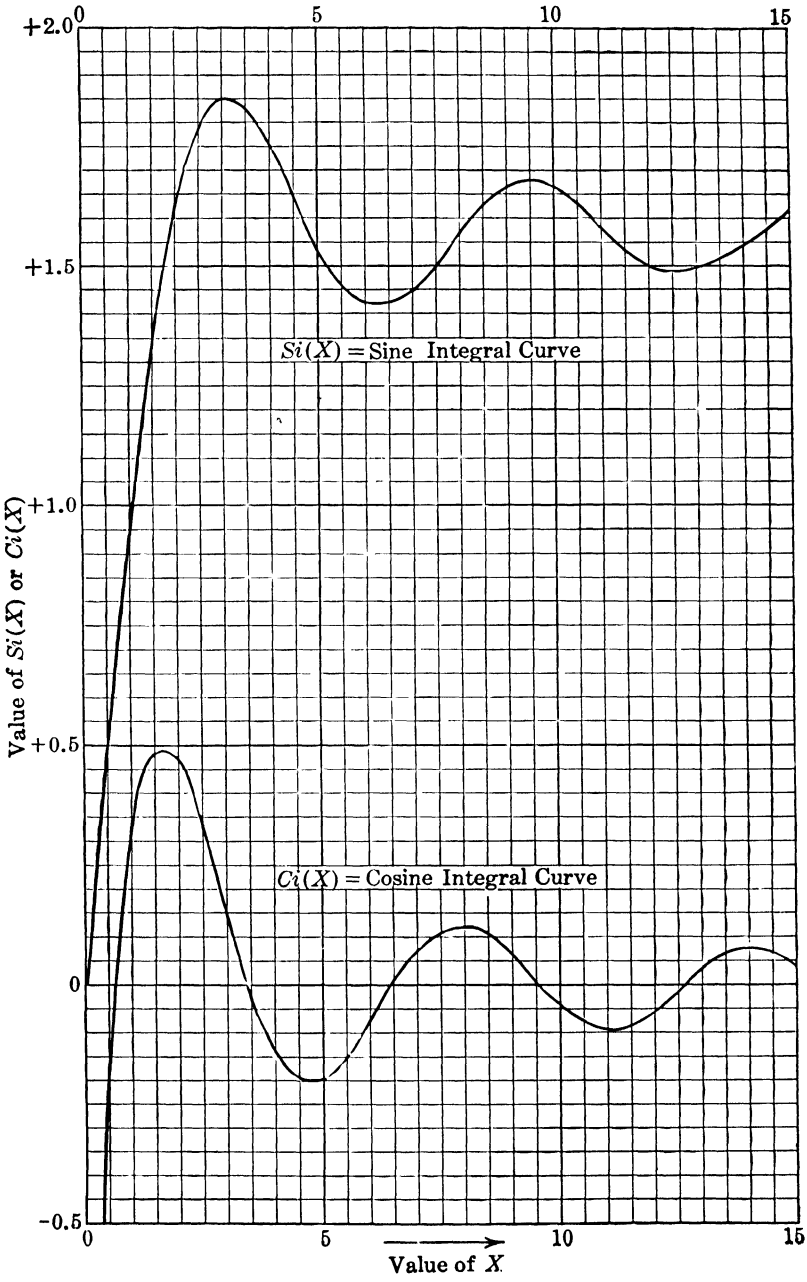
b. Currents and Voltages:

When it is necessary to distinguish between components of current and voltage encountered in electron tube circuits the symbols can be used as illustrated for the plate currents in an electron tube.



11. CURVES AND TABLES OF SINE AND COSINE INTEGRALS

a. **Sine and cosine integral curves:**



b. Sine and cosine integral tables:

x	Si(x)	Ci(x)	Ei(x)	Ei($-x$)
0.00	+0.000000	$-\infty$	$-\infty$	$-\infty$
0.01	+0.010000	−4.0280	−4.0179	−4.0379
0.02	+0.019999	−3.3349	−3.3147	−3.3547
0.03	+0.029998	−2.9296	−2.8991	−2.9591
0.04	+0.039996	−2.6421	−2.6013	−2.6813
0.05	+0.04999	−2.4191	−2.3679	−2.4679
0.06	+0.05999	−2.2371	−2.1753	−2.2953
0.07	+0.06998	−2.0833	−2.0108	−2.1508
0.08	+0.07997	−1.9501	−1.8669	−2.0269
0.09	+0.08996	−1.8328	−1.7387	−1.9187
0.10	+0.09994	−1.7279	−1.6228	−1.8229
0.11	+0.10993	−1.6331	−1.5170	−1.7371
0.12	+0.11990	−1.5466	−1.4193	−1.6595
0.13	+0.12988	−1.4672	−1.3287	−1.5889
0.14	+0.13985	−1.3938	−1.2438	−1.5241
0.15	+0.14981	−1.3255	−1.1641	−1.4645
0.16	+0.15977	−1.2618	−1.0887	−1.4092
0.17	+0.16973	−1.2020	−1.0172	−1 3578
0.18	+0.1797	−1.1457	−0.9491	−1.3098
0.19	+0.1896	−1.0925	−0.8841	−1.2649
0.20	+0.1996	−1.0422	−0.8218	−1.2227
0.21	+0.2095	−0.9944	−0.7619	−1.1829
0.22	+0.2194	−0.9490	−0.7042	−1.1454
0 23	+0.2293	−0.9057	−0.6485	−1.1099
0.24	+0.2392	−0.8643	−0.5947	−1.0762
0.25	+0.2491	−0.8247	−0.5425	−1.0443
0.26	+0.2590	−0.7867	−0.4919	−1.0139
0.27	+0.2689	−0.7503	−0.4427	−0.9849
0.28	+0.2788	−0.7153	−0.3949	−0.9573
0.29	+0.2886	−0.6816	−0.3482	−0.9309
0.30	+0.2985	−0.6492	−0.3027	−0.9057
0.31	+0.3083	−0.6179	−0.2582	−0.8815
0.32	+0.3182	−0.5877	−0.2147	−0.8583
0.33	+0.3280	−0.5585	−0.17210	−0.8361
0.34	+0.3378	−0.5304	−0.13036	−0.8147
0.35	+0.3476	−0.5031	−0.08943	−0.7942
0.36	+0.3574	−0.4767	−0.04926	−0.7745
0.37	+0.3672	−0.4511	−0.00979	−0.7554
0.38	+0.3770	−0.4263	+0.02901	−0.7371
0.39	+0.3867	−0.4022	+0.06718	−0.7194
0.40	+0.3965	−0.3788	+0.10477	−0.7024
0.41	+0.4062	−0.3561	+0.14179	−0.6859
0.42	+0.4159	−0.3341	+0.17828	−0.6700
0.43	+0.4256	−0.3126	+0.2143	−0.6546
0.44	+0.4353	−0.2918	+0.2498	−0.6397
0.45	+0.4450	−0.2715	+0.2849	−0.6253
0.46	+0.4546	−0.2517	+0.3195	−0.6114
0.47	+0.4643	−0.2325	+0.3537	−0.5979
0.48	+0.4739	−0.2138	+0.3876	−0.5848
0.49	+0.4835	−0.1956	+0.4211	−0.5721

x	Si(x)	Ci(x)	Ei(x)	Ei($-x$)
0.50	+0.4931	−0.17778	+0.4542	−0.5598
0.51	+0.5027	−0.16045	+0.4870	−0.5478
0.52	+0.5123	−0.14355	+0.5195	−0.5362
0.53	+0.5218	−0.12707	+0.5517	−0.5250
0.54	+0.5313	−0.11099	+0.5836	−0.5140
0.55	+0.5408	−0.09530	+0.6153	−0.5034
0.56	+0.5503	−0.07999	+0.6467	−0.4930
0.57	+0.5598	−0.06504	+0.6778	−0.4830
0.58	+0.5693	−0.05044	+0.7087	−0.4732
0.59	+0.5787	−0.03619	+0.7394	−0.4636
0.60	+0.5881	−0.02227	+0.7699	−0.4544
0.61	+0.5975	−0.00867	+0.8002	−0.4454
0.62	+0.6069	+0.00461	+0.8302	−0.4366
0.63	+0.6163	+0.01758	+0.8601	−0.4280
0.64	+0.6256	+0.03026	+0.8898	−0.4197
0.65	+0.6349	+0.04265	+0.9194	−0.4115
0.66	+0.6442	+0.05476	+0.9488	−0.4036
0.67	+0.6535	+0.06659	+0.9780	−0.3959
0.68	+0.6628	+0.07816	+1.0071	−0.3883
0.69	+0.6720	+0.08946	+1.0361	−0.3810
0.70	+0.6812	+0.10051	+1.0649	−0.3738
0.71	+0.6904	+0.11132	+1.0936	−0.3668
0.72	+0.6996	+0.12188	+1.1222	−0.3599
0.73	+0.7087	+0.13220	+1.1507	−0.3532
0.74	+0.7179	+0.14230	+1.1791	−0.3467
0.75	+0.7270	+0.15216	+1.2073	−0.3403
0.76	+0.7360	+0.16181	+1.2355	−0.3341
0.77	+0.7451	+0.17124	+1.2636	−0.3280
0.78	+0.7541	+0.18045	+1.2916	−0.3221
0.79	+0.7631	+0.18945	+1.3195	−0.3163
0.80	+0.7721	+0.19823	+1.3474	−0.3106
0.81	+0.7811	+0.20681	+1.3752	−0.3050
0.82	+0.7900	+0.21519	+1.4029	−0.2996
0.83	+0.7989	+0.22337	+1.4306	−0.2943
0.84	+0.8078	+0.23136	+1.4582	−0.2891
0.85	+0.8166	+0.23916	+1.4857	−0.2840
0.86	+0.8254	+0.24677	+1.5132	−0.2790
0.87	+0.8342	+0.25420	+1.5407	−0.2742
0.88	+0.8430	+0.26145	+1.5681	−0.2694
0.89	+0.8518	+0.26853	+1.5955	−0.2647
0.90	+0.8605	+0.27543	+1.6228	−0.2602
0.91	+0.8692	+0.28217	+1.6501	−0.2557
0.92	+0.8778	+0.28874	+1.6774	−0.2513
0.93	+0.8865	+0.29515	+1.7047	−0.2470
0.94	+0.8951	+0.30140	+1.7319	−0.2429
0.95	+0.9036	+0.30749	+1.7591	−0.2387
0.96	+0.9122	+0.31343	+1.7864	−0.2347
0.97	+0.9207	+0.31922	+1.8136	−0.2308
0.98	+0.9292	+0.32486	+1.8407	−0.2269
0.99	+0.9377	+0.33036	+1.8679	−0.2231
1.00	+0.9461	+0.33571	+1.8951	−0.2194

x	Si(x)	Ci(x)	Ei(x)	Ei($-x$)
1.0	+0.9461	+0.3374	+1.8951	−0.2194
1.1	+1.0287	+0.3849	+2.1674	−0.1860
1.2	+1.1080	+0.4205	+2.4421	−0.1584
1.3	+1.1840	+0.4457	+2.7214	−0.1355
1.4	+1.2562	+0.4620	+3.0072	−0.1162
1.5	+1.3247	+0.4704	+3.3013	−0.1000
1.6	+1.3892	+0.4717	+3.6053	−0.08631
1.7	+1.4496	+0.4670	+3.9210	−0.07465
1.8	+1.5058	+0.4568	+4.2499	−0.06471
1.9	+1.5578	+0.4419	+4.5937	−0.05620
2.0	+1.6054	+0.4230	+4.9542	−0.04890
2.1	+1.6487	+0.4005	+5.3332	−0.04261
2.2	+1.6876	+0.3751	+5.7326	−0.03719
2.3	+1.7222	+0.3472	+6.1544	−0.03250
2.4	+1.7525	+0.3173	+6.6007	−0.02844
2.5	+1.7785	+0.2859	+7.0738	−0.02491
2.6	+1.8004	+0.2533	+7.5761	−0.02185
2.7	+1.8182	+0.2201	+8.1103	−0.01918
2.8	+1.8321	+0.1865	+8.6793	−0.01686
2.9	+1.8422	+0.1529	+9 2860	−0 01482
3.0	+1.8487	+0.1196	+9 9338	−0.01304
3.1	+1.8517	+0.08699	+10.6263	−0.01149
3.2	+1.8514	+0.05526	+11 3673	−0.01013
3.3	+1.8481	+0.02468	+12 1610	−0 0^{2}8939
3.4	+1.8419	−0.004518	+13.0121	−0.0^{2}7890
3.5	+1.8331	−0.03213	+13.9254	−0.0^{2}6970
3.6	+1.8219	−0.05797	+14 9063	−0.0^{2}6160
3.7	+1.8086	−0.08190	+15 9606	−0.0^{2}5448
3.8	+1.7934	−0.1038	+17.0948	−0.0^{2}4820
3.9	+1.7765	−0.1235	+18.3157	−0.0^{2}4267
4.0	+1.7582	−0.1410	+19.6309	−0.0^{2}3779
4.1	+1.7387	−0.1562	+21.0485	−0.0^{2}3349
4.2	+1.7184	−0.1690	+22 5774	−0.0^{2}2969
4.3	+1.6973	−0.1795	+24 2274	−0.0^{2}2633
4.4	+1.6758	−0.1877	+26.0090	−0.0^{2}2336
4.5	+1.6541	−0.1935	+27.9337	−0.0^{2}2073
4.6	+1.6325	−0.1970	+30.0141	−0.0^{2}1841
4.7	+1.6110	−0.1984	+32.2639	−0.0^{2}1635
4.8	+1.5900	−0.1976	+34.6979	−0.0^{2}1453
4.9	+1.5696	−0.1948	+37.3325	−0.0^{2}1291
5.0	+1.5499	−0.1900	+40.1853	−0.0^{2}1148
6	+1.4247	−0.06806	+85.9898	−0.0^{3}3601
7	+1.4546	+0.07670	+191.505	−0.0^{3}1155
8	+1.5742	+0.1224	+440.380	−0.0^{4}3767
9	+1.6650	+0.05535	+1037.88	−0.0^{4}1245
10	+1.6583	−0.04546	+2492.23	−0.0^{5}4157
11	+1.5783	−0.08956	+6071.41	−0.0^{5}1400
12	+1.5050	−0.04978	+14959.5	−0.0^{6}4751
13	+1.4904	+0.02676	+37197.7	−0.0^{6}1622
14	+1.5562	+0.06940	+93192.5	−0.0^{7}5566
15	+1.6182	+0.04628	+23495.6	−0.0^{7}1918

x	Si(x)	Ci(x)	x	Si(x)	Ci(x)
20	+1.5482	+0.04442	140	+1.5722	+0.007011
25	+1.5315	−0.00685	150	+1.5662	−0.004800
30	+1.5668	−0.03303	160	+1.5769	+0.001409
35	+1.5969	−0.01148	170	+1.5653	+0.002010
40	+1.5870	+0.01902	180	+1.5741	−0.004432
45	+1.5587	+0.01863	190	+1.5704	+0.005250
50	+1.5516	−0.00563	200	+1.5684	−0.004378
55	+1.5707	−0.01817	300	+1.5709	−0.003332
60	+1.5867	−0.00481	400	+1.5721	−0.002124
65	+1.5775	+0.01285	500	+1.5726	−0.0009320
70	+1.5616	+0.01092	600	+1.5725	+0.0000764
75	+1.5586	−0.00533	700	+1.5720	+0 0007788
80	+1.5723	−0.01240	800	+1.5714	+0.001118
85	+1 5824	−0.001935	900	+1.5707	+0.001109
90	+1.5757	+0.009986	10^3	+1.5702	+0.000826
95	+1.5630	+0.007110	10^4	+1.5709	−0.0000306
100	+1.5622	−0.005149	10^5	+1.5708	+0.0000004
110	+1.5799	−0.000320	10^6	+1.5708	−0.0000004
120	+1.5640	+0.004781	10^7	+1.5708	+0.0
130	+1.5737	−0.007132	∞	$\frac{1}{2}\pi$	0.0

$\frac{x}{\pi}$	Max. Min. (Ci x)	$\frac{x}{\pi}$	Max. Min. (Si x)
0.5	+0.472 00	1	+0.281 14
1.5	−0.198 41	2	−0.152 64
2.5	+0.123 77	3	+0.103 96
3.5	−0.089 564	4	−0.078 635
4.5	+0.070 065	5	+0.063 168
5.5	−0.057 501	6	−0.052 762
6.5	+0.048 742	7	+0.045 289
7.5	−0.042 292	8	−0.039 665
8 5	+0.037 345	9	+0.035 280
9.5	−0.033 433	10	−0.031 767
10.5	+0.030 260	11	+0.028 889
11.5	−0.027 637	12	−0.026 489
12.5	+0.025 432	13	+0.024 456
13.5	−0.023 552	14	−0.022 713
14.5	+0.021 931	15	+0.021 201
15.5	−0.020 519		

12. FACTORIALS OF NUMBERS FROM 1 TO 20 AND THEIR RECIPROCALS

n	$n!$ or $\lfloor\underline{n}$	$\frac{1}{n!}$ or $\frac{1}{\lfloor\underline{n}}$
1	1	1.
2	2	5. 10^{-1}
3	6	1.6666 6667 10^{-1}
4	2.4 10	4.1666 6667 10^{-2}
5	1.2 10^{2}	8.3333 3333 10^{-3}
6	7.2 10^{2}	1.3888 8889 10^{-4}
7	5.04 10^{3}	1.9841 2698 10^{-4}
8	4.032 10^{4}	2.4857 0718 10^{-5}
9	3.6288 10^{5}	2.7557, 3192 10^{-6}
10	3.6288 10^{6}	2.7557 3192· 10^{-7}
11	3.9916 8 10^{7}	2.5052 1084 10^{-8}
12	4.7900 16 10^{8}	2.0876 7570 10^{-9}
13	6.2270 208 10^{9}	1.6059 0438 10^{-10}
14	8.7178 2912 10^{10}	1.1470 7456 10^{-11}
15	1.3076 7437 10^{12}	7.6471 6373 10^{-13}
16	2.0922 7899 10^{13}	4.7794 7733 10^{-14}
17	3.5568 7428 10^{14}	2.8114 5725 10^{-15}
18	6.4023 7371 10^{15}	1.5619 2069 10^{-16}
19	1.2164 5100 10^{17}	8.2206 3525 10^{-18}
20	2.4329 0201 10^{18}	4.1103 1761 10^{-19}

13. USEFUL CONSTANTS

(1) $\epsilon = 2.71828183$

(2) $\frac{1}{\epsilon} = 0.36787944$

(3) $\log \epsilon = 0.43429448$

(4) $\log \log \epsilon = 9.63778431 - 10$

(5) $\pi = 3.14159265$

(6) $\frac{1}{\pi} = 0.31830989$

(7) $\pi^2 = 9.86960440$

(8) $\sqrt{\pi} = 1.77245385$

(9) $\log \pi = 0.49714987$

(10) $\ln \pi = 1.14472989$

(11) $\log 2 = 0.30103032$

(12) $\log 3 = 0.47712125$

(13) $\ln 2 = 0.69314718$

(14) $\ln 3 = 1.09861229$

(15) $\ln 10 = 2.30258509$

(16) $\gamma = 0.57721566$
$= \text{Euler's constant}$

(17) $\frac{N_{db}}{\alpha} = \frac{\text{decibels}}{\text{nepers}} = 8.68588964$

(18) $\frac{\alpha}{N_{db}} = \frac{\text{nepers}}{\text{decibels}} = 0.11512925$

(19) $1 \text{ radian} = \frac{180}{\pi} = 57°17'44.8''$
$= 57°29577951$

(20) $1° = \frac{\pi}{180} = 0.01745329 \text{ radians}$

(21) $\sqrt{2} = 1.41421356$

(22) $\sqrt{3} = 1.73205081$

14. ALGEBRAIC FORMULAS

a. **Miscellaneous Formulas:**

(1) $a + b = b + a$

(2) $(a + b) + c = a + (b + c)$

(3) $a + c = b + d$, if $a = b$ and $c = d$

(4) $ab = ba$

(5) $(ab)c = a(bc)$

(6) $a(b + c) = ab + ac$

(7) $ac = bd$, if $a = b$ and $c = d$

(8) $a + (-b) = a - b$

(9) $a - (-b) = a + b$

(10) $0 \cdot a = 0$

(11) $\frac{0}{a} = 0$

(12) $\infty \cdot a = \infty$

(13) $\frac{\infty}{a} = \infty$

(14) $0 \cdot \infty = \text{indeterminate}$

(15) $\frac{0}{0} = \text{indeterminate}$

(16) $\frac{\infty}{\infty} = \text{indeterminate}$

(17) $a(-b) = -ab$

(18) $(-a)(-b) = ab$

(19) $-(a - b + c) = -a + b - c$

(20) $\frac{a}{b} = \frac{ac}{bc}$

(21) $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$

(22) $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$

(23) $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

(24) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

(25) $\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$

(26) $\frac{\frac{a}{b}}{\frac{c}{d}} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$

(27) $(a^m)(a^n) = a^{m+n}$

(28) $(a^m)^n = a^{mn}$

(29) $(abc)^m = a^m b^m c^m$

(30) $\left(\frac{c}{b}\right)^m = \frac{a^m}{b^m}$

(31) $\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}$

(32) $a^0 = 1$

(33) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(34) $a^{-m} = \frac{1}{a^m}$

(35) $(a+b)(a+b) = a^2 + 2ab + b^2$

(36) $(a+b)(a-b) = a^2 - b^2$

(37) $(a-b)(a-b) = a^2 - 2ab + b^2$

(38) $(a+b)(c-d) = ac + bc - ad - bd$

(39) Factorial number $n! = \lfloor n = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1)n$

b. Simultaneous equations:

1. The solution of the two simultaneous equations

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

by determinants is

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

2. The solution of three simultaneous equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

by determinants is

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

c. **Quadratic equations:**

For the quadratic equation

$$ax^2 + bx + c = 0$$

the solution by the quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the discriminant $b^2 - 4ac$ is positive, zero, or negative, the roots are accordingly real and unequal, real and equal, or unequal and complex numbers.

d. **Binomial theorem** (n being a positive integer):

$$(a \pm b)^n = a^n \pm na^{n-1}b + \frac{n(n-1)}{\underline{|2}} a^{n-2}b^2 \pm \frac{n(n-1)(n-2)}{\underline{|3}} a^{n-3}b^3 + \cdots$$

e. **Trapezoidal rule:**

$$A = \Delta x \left[\tfrac{1}{2}(y_o + y_n) + \sum_{k=1}^{k=n-1} y_k \right]$$

where A = area
Δx = width of strips
y_k = height of kth strip
n = number of strips

f. **Simpson's rule:**

$$A = \frac{\Delta x}{3} \left[(y_o + y_n) + 4 \sum_{k=1}^{k=\frac{n}{2}} y_{(2k-1)} + 2 \sum_{k=1}^{k=\frac{n}{2}-1} y_{2k} \right]$$

where the letters have the same meaning as given above in the trapezoidal rule.

15. LOGARITHMIC FORMULAS

(1) ln = natural logarithm to base ϵ
(2) log = common logarithm to base 10
(3) $\log 1 = 0 \qquad \ln 1 = 0$
(4) $\log 10 = 1 \qquad \ln \epsilon = 1$
(5) $\log ab = \log a + \log b \qquad \ln ab = \ln a + \ln b$
(6) $\log a^n = n \log a \qquad \ln a^n = n \ln a$
(7) $\log a^{\frac{1}{n}} = \frac{1}{n} \log a \qquad \ln a^{\frac{1}{n}} = \frac{1}{n} \ln a$
(8) If $y = \log x$, $10^y = x$ — If $y = \ln x$, $\epsilon^y = x$
(9) $\log x = \log \epsilon \ln x = 0.4343 \ln x$
(10) $\ln x = \ln 10 \log x = 2.3026 \log x$
(11) The characteristic of a number is the whole number part of the common logarithm and is the exponent of 10 when the number is written in scientific notation.
(12) The mantissa of a number is the decimal part of the common logarithm found in the tables.
(13) Example of characteristic and mantissa

$$\log (3.1623)10^2 = 2.500$$

where 2 = the characteristic, from exponent of 10
0.5 = the mantissa, from logarithm tables

Thus $316.23 = 10^{2.5}$

16. COMPLEX QUANTITIES

(1) $j = \sqrt{-1}$ or $j^2 = -1$
(2) $a + jb = \sqrt{a^2 + b^2}\,(\cos \alpha + j \sin \alpha)$
$= C\epsilon^{j\alpha} = C\underline{/\alpha}$
where $C = \sqrt{a^2 + b^2}$
$\tan \alpha = \frac{b}{a} = \frac{\sin \alpha}{\cos \alpha}$
(3) $(a + jb) + (c + jd) = (a + c) + j(b + d)$
(4) $(a + jb) - (c + jd) = (a - c) + j(b - d)$
(5) $(a + jb)(c + jd) = (ac - bd) + j(bc + ad)$
(6) $(a + jb)(c - jd) = (ac + bd) + j(bc - ad)$
(7) $\frac{a + jb}{c + jd} = \frac{a + jb}{c + jd} \cdot \frac{c - jd}{c - jd} = \frac{ac + bd}{c^2 + d^2} + j\frac{bc - ad}{c^2 + d^2}$
(8) $\frac{a + jb}{c + jd} = \frac{C\epsilon^{j\alpha}}{E\epsilon^{j\beta}} = \frac{C}{E}\epsilon^{j(\alpha - \beta)} = \frac{C}{E}\cos(\alpha - \beta) + j\frac{C}{E}\sin(\alpha - \beta)$
where $C = \sqrt{a^2 + b^2}$; $\tan \alpha = \frac{b}{a}$
$E = \sqrt{c^2 + d^2}$; $\tan \beta = \frac{d}{c}$
(9) If $a + jb = c + jd$
then $a = c$
and $b = d$

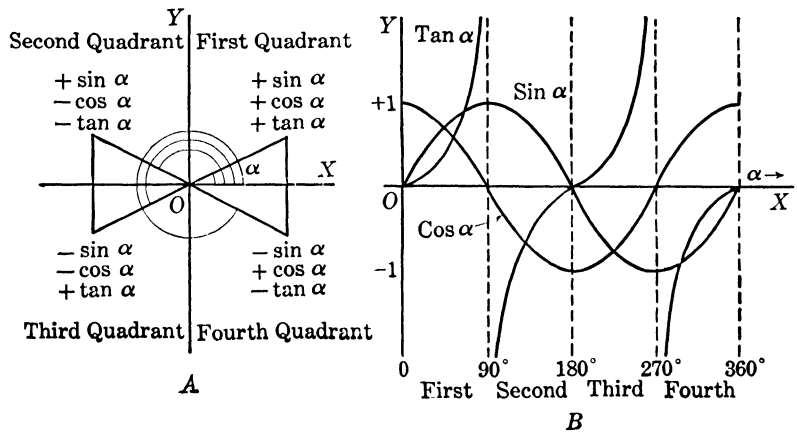
17. TRIGONOMETRIC CHARTS AND FORMULAS

a. **Chart of fraction values for trigonometric functions:**

Degrees	Sine	Cosine	Tangent	Cotangent
0	0	1	0	∞
30	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$
45	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1
60	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$
90	1	0	∞	0
120	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$
135	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1
150	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$
180	0	-1	0	∞

b. **Chart for reducing angles:**

Angle	Sine	Cosine	Tangent	Cotangent
$\pm\alpha$	$\pm\sin\alpha$	$+\cos\alpha$	$\pm\tan\alpha$	$\pm\cot\alpha$
$90° \pm \alpha$	$+\cos\alpha$	$\mp\sin\alpha$	$\mp\operatorname{ctn}\alpha$	$\mp\tan\alpha$
$180° \pm \alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\tan\alpha$	$\pm\cot\alpha$
$270° \pm \alpha$	$-\cos\alpha$	$\pm\sin\alpha$	$\mp\operatorname{ctn}\alpha$	$\mp\tan\alpha$
$360° \pm \alpha$	$\pm\sin\alpha$	$+\cos\alpha$	$\pm\tan\alpha$	$\pm\cot\alpha$

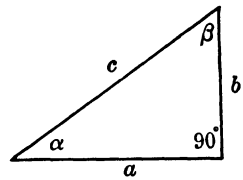


Algebraic signs and graphs of trigonometric functions in the various quadrants.

c. Trigonometric functions in terms of each other:

Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
$\sin \alpha$	$\sqrt{1 - \cos^2 \alpha}$	$\frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}$	$\frac{1}{\sqrt{1 + \cot^2 \alpha}}$	$\frac{\sqrt{\sec^2 \alpha - 1}}{\sec \alpha}$	$\frac{1}{\csc \alpha}$
$\sqrt{1 - \sin^2 \alpha}$	$\cos \alpha$	$\frac{1}{\sqrt{1 + \tan^2 \alpha}}$	$\frac{\cot \alpha}{\sqrt{1 + \cot^2 \alpha}}$	$\frac{1}{\sec \alpha}$	$\frac{\sqrt{\csc^2 \alpha - 1}}{\csc \alpha}$
$\frac{\sin \alpha}{\sqrt{1 - \sin^2 \alpha}}$	$\frac{\sqrt{1 - \cos^2 \alpha}}{\cos \alpha}$	$\tan \alpha$	$\frac{1}{\cot \alpha}$	$\sqrt{\sec^2 \alpha - 1}$	$\frac{1}{\sqrt{\csc^2 \alpha - 1}}$
$\frac{\sqrt{1 - \sin^2 \alpha}}{\sin \alpha}$	$\frac{\cos \alpha}{\sqrt{1 - \cos^2 \alpha}}$	$\frac{1}{\tan \alpha}$	$\cot \alpha$	$\frac{1}{\sqrt{\sec^2 \alpha - 1}}$	$\sqrt{\csc^2 \alpha - 1}$
$\frac{1}{\sqrt{1 - \sin^2 \alpha}}$	$\frac{1}{\cos \alpha}$	$\sqrt{1 + \tan^2 \alpha}$	$\frac{\sqrt{1 + \cot^2 \alpha}}{\cot \alpha}$	$\sec \alpha$	$\frac{\csc \alpha}{\sqrt{\csc^2 \alpha - 1}}$
$\frac{1}{\sin \alpha}$	$\frac{1}{\sqrt{1 - \cos^2 \alpha}}$	$\frac{\sqrt{1 + \tan^2 \alpha}}{\tan \alpha}$	$\sqrt{1 + \cot^2 \alpha}$	$\frac{\sec \alpha}{\sqrt{\sec^2 \alpha - 1}}$	$\csc \alpha$

d. Trigonometric formulas:



Right triangle for use with trigonometric formulas.

(1) $\sin \alpha = \frac{b}{c} = \cos \beta$

(2) $\cos \alpha = \frac{a}{c} = \sin \beta$

(3) $\tan \alpha = \frac{b}{a} = \cot \beta$

(4) $\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\cos \beta}{\sin \beta}$

(5) $\sin \alpha = \frac{\epsilon^{j\alpha} - \epsilon^{-j\alpha}}{2j}$

(6) $\cos \alpha = \frac{\epsilon^{j\alpha} + \epsilon^{-j\alpha}}{2}$

(7) $\tan \alpha = -j \frac{\epsilon^{j\alpha} - \epsilon^{-j\alpha}}{\epsilon^{j\alpha} + \epsilon^{-j\alpha}}$

(8) $\sin (-\alpha) = -\sin \alpha$

(9) $\cos (-\alpha) = \cos \alpha$

(10) $\tan (-\alpha) = -\tan \alpha$

(11) $\sin \left(\frac{\pi}{2} \pm \alpha\right) = \cos \alpha$

(12) $\cos \left(\frac{\pi}{2} \pm \alpha\right) = \mp \sin \alpha$

(13) $\tan \left(\frac{\pi}{2} + \alpha\right) = \mp \cot \alpha$

(14) $\sin (\pi \pm \alpha) = \mp \sin \alpha$

(15) $\cos (\pi \pm \alpha) = -\cos \alpha$

(16) $\tan (\pi \pm \alpha) = \mp \tan \alpha$

(17) $\sin \frac{\alpha}{2} = \sqrt{\frac{1}{2}(1 - \cos \alpha)}$

(18) $\cos \frac{\alpha}{2} = \sqrt{\frac{1}{2}(1 + \cos \alpha)}$

(19) $\tan \frac{\alpha}{2} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$

(20) $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

(21) $\cos 2\alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$

(22) $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

(23) $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$

(24) $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$

(25) $\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$

(26) $\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$

(27) $\sin^2 \alpha + \cos^2 \alpha = 1$

(28) $\sin^3 \alpha = \frac{3 \sin \alpha - \sin 3\alpha}{4}$

(29) $\cos^3 \alpha = \frac{3 \cos \alpha + \cos 3\alpha}{4}$

(30) $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

(31) $\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

(32) $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

(33) $\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

(34) $\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

(35) $\tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

(36) $\sin (\alpha + \beta) + \sin (\alpha - \beta) = 2 \sin \alpha \cos \beta$

(37) $\sin (\alpha + \beta) - \sin (\alpha - \beta) = 2 \cos \alpha \sin \beta$

(38) $\cos (\alpha + \beta) + \cos (\alpha - \beta) = 2 \cos \alpha \cos \beta$

(39) $\cos (\alpha - \beta) - \cos (\alpha + \beta) = 2 \sin \alpha \cos \beta$

18. EXPONENTIAL AND HYPERBOLIC FORMULAS

***a.* Exponential formulas:**

(1) $\epsilon^{j\beta} = \cos\beta + j\sin\beta$

(2) $\epsilon^{-j\beta} = \cos\beta - j\sin\beta$

(3) $\epsilon^{\alpha} = \cosh\alpha + \sinh\alpha$

(4) $\epsilon^{-\alpha} = \cosh\alpha - \sinh\alpha$

(5) $\epsilon^{\alpha+j\beta} = \epsilon^{\alpha}\cos\beta + j\epsilon^{\alpha}\sin\beta$
$= (\cosh\alpha\cos\beta + \sinh\alpha\cos\beta)$
$+ j(\cosh\alpha\sin\beta) + \sinh\alpha\sin\beta)$

***b.* Relation between hyperbolic and circular functions:**

(1) $\sinh j\beta = j\sin\beta$

(2) $\cosh j\beta = \cos\beta$

(3) $\sin j\beta = j\sinh\beta$

(4) $\cos j\beta = \cosh\beta$

***c.* Hyperbolic functions:**

(1) $\sinh\beta = \dfrac{\epsilon^{\beta} - \epsilon^{-\beta}}{2}$

(2) $\cosh\beta = \dfrac{\epsilon^{\beta} + \epsilon^{-\beta}}{2}$

(3) $\tanh\beta = \dfrac{\epsilon^{\beta} - \epsilon^{-\beta}}{\epsilon^{\beta} + \epsilon^{-\beta}}$

(4) $\sinh(-\beta) = -\sinh\beta$

(5) $\cosh(-\beta) = \cosh\beta$

(6) $\cosh^2\beta - \sinh^2\beta = 1$

(7) $\sinh(\alpha + \beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta$

(8) $\sinh(\alpha - \beta) = \sinh\alpha\cosh\beta - \cosh\alpha\sinh\beta$

(9) $\cosh(\alpha + \beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta$

***d.* Circular and hyperbolic functions of complex quantities:**

(1) $\sinh(\alpha \pm j\beta) = \sinh\alpha\cosh j\beta \pm \cosh\alpha\sinh j\beta$
$= \sinh\alpha\cos\beta \pm j\cosh\alpha\sin\beta$
$= \sqrt{\sinh^2\alpha + \sin^2\beta} \,/\, \pm\tan^{-1}\dfrac{\tan\beta}{\tanh\alpha}$

(2) $\cosh(\alpha \pm j\beta) = \cosh\alpha\cosh j\beta \pm \sinh\alpha\sinh j\beta$
$= \cosh\alpha\cos\beta \pm j\sinh\alpha\sin\beta$
$= \sqrt{\sinh^2\alpha + \cos^2\beta} \,/\, \pm\tan^{-1}(\tan\beta\tanh\alpha)$

(3) $\sin(\alpha \pm j\beta) = \sin\alpha\cos j\beta \pm \cos\alpha\sin j\beta$
$= \sin\alpha\cosh\beta \pm j\cos\alpha\sinh\beta$
$= \sqrt{\sin^2\alpha + \sinh^2\beta} \,/\, \pm\tan^{-1}\dfrac{\tan\beta}{\tan\alpha}$

(4) $\cos(\alpha \pm j\beta) = \cos\alpha\cos j\beta \mp \sin\alpha\sin j\beta$
$= \cos\alpha\cosh\beta \mp j\sin\alpha\sinh\beta$
$= \sqrt{\cos^2\alpha + \sinh^2\beta} \,/\, \mp\tan^{-1}(\tanh\beta\tan\alpha)$

19. STANDARD ELEMENTARY CALCULUS FORMULAS

Derivative $\frac{dy}{dx}$	Function y	Integral $\int y\,dx$
	Algebraic	
0	0	C
0	1	$x + C$
0	a	$ax + C$
1	x	$\frac{1}{2}x^2 + C$
a	ax	$\frac{1}{2}ax^2 + C$
$2x$	x^2	$\frac{1}{3}x^3 + C$
nx	x^n	$\frac{1}{n+1}x^{n+1} + C$
$-x^{-2}$	x^{-1}	$\ln x + C$
$\frac{du}{dx} + \frac{dv}{dx}$	$u + v$	$\int u\,dx + \int v\,dx$
$u\frac{dv}{dx} + v\frac{du}{dx}$	$u\,v$	No general form
$\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$	$\frac{u}{v}$	No general form
$\frac{du}{dv}$	u	$\int u\,dv = uv - \int v\,du + C$
$-\frac{1}{(x+a)^2}$	$\frac{1}{x+a}$	$\ln(x+a) + C$
$-\frac{x}{(a^2+x^2)^{3/2}}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln(x + \sqrt{a^2+x^2}) + C$
$\mp\frac{b}{(a \pm bx)^2}$	$\frac{1}{a \pm bx}$	$\pm\frac{1}{b}\ln(a+bx) + C$
$\frac{-3a^2x}{(a^2+x^2)^{5/2}}$	$\frac{a^2}{(a^2+x^2)^{3/2}}$	$\frac{x}{\sqrt{a^2+x^2}} + C$
	Transcendental	
ϵ^x	ϵ^x	$\epsilon^x + C$
$a\epsilon^{ax}$	ϵ^{ax}	$\frac{1}{a}\epsilon^{ax} + C$
$a^x \ln a$	a^x	$\frac{a^x}{\ln a} + C$
$ba^{bx}\ln a$	a^{bx}	$\frac{a^{bx}}{b\ln a} + C$
x^{-1}	$\ln x$	$x(\ln x - 1) + C$
$0.4343x^{-1}$	$\log x$	$0.4343(\ln x - 1) + C$
$\cos x$	$\sin x$	$-\cos x + C$

19. STANDARD ELEMENTARY CALCULUS FORMULAS.—*(Continued)*

Derivative $\frac{dy}{dx}$	Function y	Integral $\int y\,dx$
$a \cos ax$	$\sin ax$	$-\frac{1}{a}\cos ax + C$
$-\sin x$	$\cos x$	$\sin x + C$
$-a \sin ax$	$\cos ax$	$\frac{1}{a}\sin ax + C$
$\sec^2 x$	$\tan x$	$-\ln \cos x + C$
$a \sec^2 ax$	$\tan ax$	$-\frac{1}{a}\ln \cos ax + C$
$\cosh x$	$\sinh x$	$\cosh x + C$
$a \cosh ax$	$\sinh ax$	$\frac{1}{a}\cosh ax + C$
$\sinh x$	$\cosh x$	$\sinh x + C$
$a \sinh x$	$\cosh ax$	$\frac{1}{a}\sinh ax + C$
$\operatorname{sech}^2 x$	$\tanh x$	$\ln \cosh x + C$
$a \operatorname{sech}^2 ax$	$\tanh ax$	$-\frac{1}{a}\ln \cosh ax + C$

20. OTHER USEFUL INTEGRALS

$$\int x\epsilon^{\alpha x}\,dx = \frac{1}{\alpha^2}\epsilon^{\alpha x}(\alpha x - 1)$$

$$\int \epsilon^{\alpha t}\sin(\omega t + \psi)\,dt = \frac{\epsilon^{\alpha t}}{\alpha^2 + \omega^2}[\alpha \sin(\omega t + \psi) - \omega \cos(\omega t + \psi)]$$

$$\int \epsilon^{\alpha t}\cos(\omega t + \psi)\,dt = \frac{\epsilon^{\alpha t}}{\alpha^2 + \omega^2}[\alpha \cos(\omega t + \psi) + \omega \sin(\omega t + \psi)]$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin x}{x}\,dx$$

$$\mathrm{Ci}(x) = \int_\infty^x \frac{\cos x}{x}\,dx$$

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\sin\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cos(x\cos\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\cos(x\cos\theta)\,d\theta$$

$$= \frac{j}{2\pi}\int_0^{2\pi}\sin(x\sin\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\epsilon^{jx\sin\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\epsilon^{jx\cos\theta}\,d\theta = \frac{1}{\pi}\int_0^{\pi}\epsilon^{jx\cos\theta}\,d\theta$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\,d\theta$$

$$= \frac{j}{\pi}\int_0^{\pi}\sin(n\theta - x\sin\theta)\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\epsilon^{j(n\theta - x\sin\theta)}\,d\theta$$

21. SERIES

a. **Exponential:**

(1) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$

(2) $a^x = 1 + x \log a + \frac{(x \log a)^2}{\lfloor 2} + \frac{(x \log a)^3}{\lfloor 3} + \cdots$

b. **Trigonometric:**

(1) $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$

(2) $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

(3) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$

c. **Hyperbolic:**

(1) $\sinh x = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \frac{x^7}{\lfloor 7} + \cdots$

(2) $\cosh x = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \frac{x^6}{\lfloor 6} + \cdots$

(3) $\tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$

d. **Logarithmic:**

$$\ln (1 + x) = x - \tfrac{1}{2} x^2 + \tfrac{1}{3} x^3 - \tfrac{1}{4} x^4 + \cdots$$

e. **Sine and cosine integral:**

$$\mathrm{Si}(x) = \int_0^x \frac{\sin x}{x}\, dx = x - \frac{x^3}{3\lfloor 3} + \frac{x^5}{5\lfloor 5} - \cdots$$

$$\mathrm{Ci}(x) = \int_\infty^x \frac{\cos x}{x}\, dx = \gamma + \ln x - \frac{x^2}{2\lfloor 2} + \frac{x^4}{4\lfloor 4} - \frac{x^6}{6\lfloor 6} + \cdots$$

f. **Bessel's:**

(1) $J_0(x) = 1 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

(2) $J_n(x) = \frac{x^n}{2^n \lfloor n} \left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2 \cdot 4(2n+2)(2n+4)} - \frac{x^6}{2 \cdot 4 \cdot 6(2n+2)(2n+4)(2n+6)} + \cdots \right]$

$$= \sum_{k=0}^{k=\infty} \frac{(-1)^k}{\lfloor k \lfloor n+k} \left(\frac{x}{2}\right)^{n+2k} \qquad (n \text{ an integer})$$

g. **Binomial:**

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{\lfloor 2} x^2 \pm \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \cdots$$

h. **Power:**

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

i. **Maclaurin's:**

$$f(x) = f(o) + xf'(o) + \frac{x^2}{\lfloor 2}f''(o) + \frac{x^3}{\lfloor 3}f'''(o) + \cdots$$

j. **Taylor's:**

(1) $f(x) = f(a) + (x - a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \frac{(x-a)^3}{\lfloor 3}f'''(a) + \cdots$

(2) $f(x + h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}f'''(x) + \cdots$

k. **Fourier's:**

(1) $f(x) = \frac{A_0}{2} + A_1 \sin x + A_2 \cos 2x + A_3 \cos 3x + \cdots$

$$+ B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x + \cdots$$

$$= \frac{A_0}{2} + \sum_{n=1}^{n=\infty} (A_n \cos nx + B_n \sin nx)$$

where $A_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx$$

(2) $f(x) = M_0 + M_1 \sin(\omega t + \psi_1) + M_2 \sin(\omega t + \psi_2) + \cdots$

$$= \sum_{n=o}^{n=\infty} M_n \sin(\omega t + \psi_n)$$

where $n = 0$

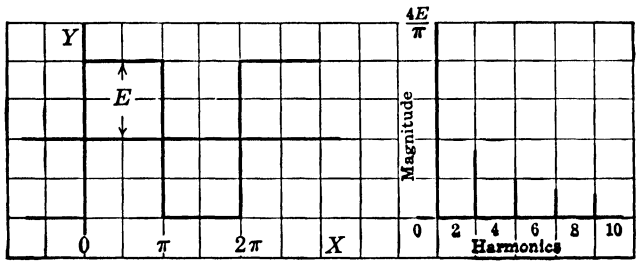
$$M_o = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

and for $n = 1, 2, 3, \cdots$

$$M_n \angle \psi_n = \frac{j}{\pi}\int_0^{2\pi} f(x)\epsilon^{-inx}\,dx$$

22. WAVEFORMS

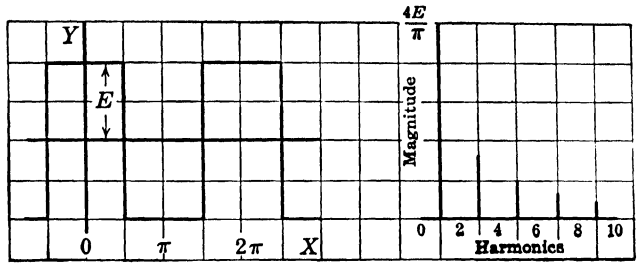
a. **Square sine wave:**



$$y = \frac{4E}{\pi}\sum_{n=1}^{n=\infty} \frac{1}{2n-1} \sin(2n-1)x$$

$$= \frac{4E}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

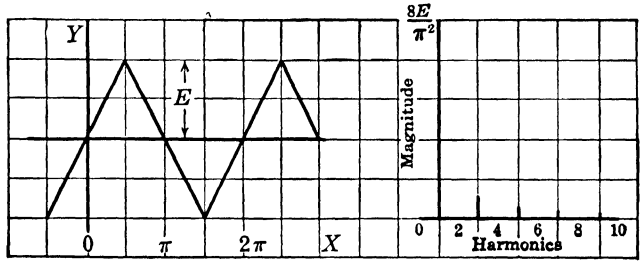
b. **Square cosine wave:**



$$y = \frac{4E}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{2n-1} \sin (2n-1)x$$

$$= \frac{4E}{\pi} \left[\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots \right]$$

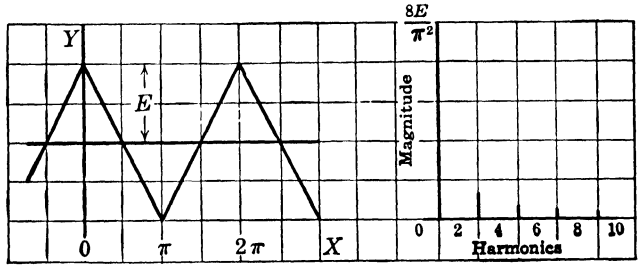
c. **Sawtooth sine wave:**



$$y = \frac{8E}{\pi^2} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin (2n-1)x$$

$$= \frac{8E}{\pi^2} \left[\sin x - \frac{1}{9} \sin 3x + \frac{1}{25} \sin 5x - \cdots \right]$$

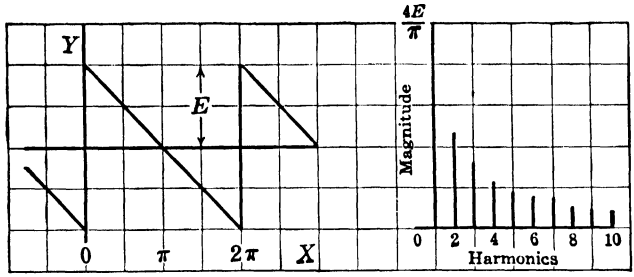
d. **Sawtooth cosine wave:**



$$y = \frac{8E}{\pi^2} \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} \cos (2n-1)x$$

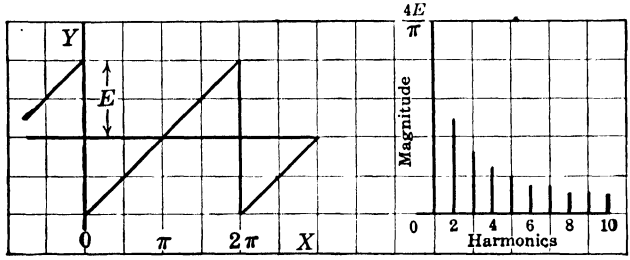
$$= \frac{8E}{\pi^2} \left[\cos x + \frac{1}{9} \cos 3x + \frac{1}{25} \cos 5x + \cdots \right]$$

e. **Positive scanning wave:**



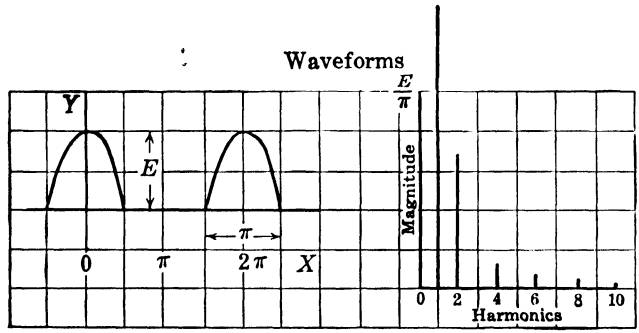
$$y = \frac{2E}{\pi} \sum_{n=1}^{n=\infty} \frac{\sin nx}{n} = \frac{2E}{\pi} \left[\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots \right]$$

f. **Negative scanning wave:**



$$y = -\frac{2E}{\pi} \sum_{n=1}^{n=\infty} \frac{\sin nx}{n} = \frac{2E}{\pi} \left[-\sin x - \frac{1}{2} \sin 2x - \frac{1}{3} \sin 3x - \cdots \right]$$

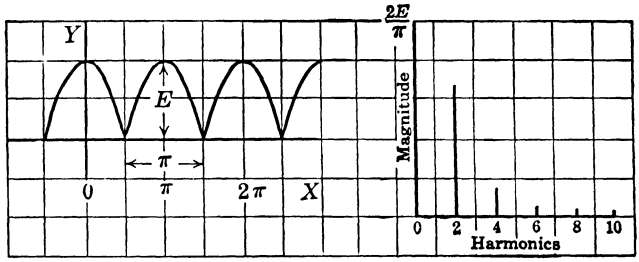
g. **Half-wave rectifier pulse wave:**



$$y = \frac{E}{\pi} + \frac{E}{2} \cos x + \frac{2E}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{4n^2 - 1} \cos 2nx$$

$$= \frac{E}{\pi} \left[1 + \frac{\pi}{2} \cos x + \frac{2}{3} \cos 2x - \frac{2}{15} \cos 4x + \cdots \right]$$

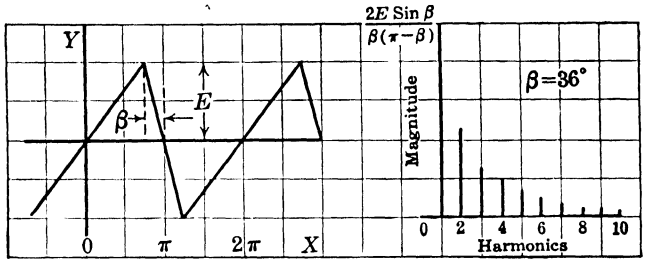
h. **Full-wave rectifier pulse wave:**



$$y = \frac{2E}{\pi} + \frac{4E}{\pi} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{4n^2 - 1} \cos 2nx$$

$$= \frac{2E}{\pi}\left[1 + \frac{2}{3}\cos 2x - \frac{2}{15}\cos 4x + \frac{2}{35}\cos 6x - \cdots\right]$$

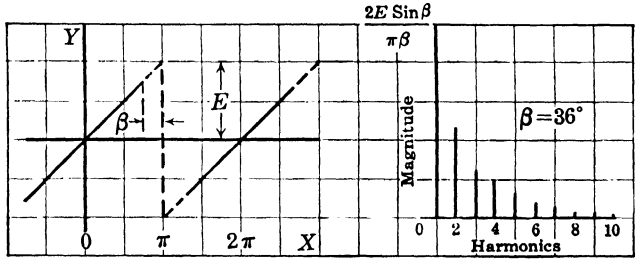
i. **Unsymmetrical sawtooth wave:**



$$y = \frac{2E}{\beta(\pi - \beta)} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n^2} \sin n\beta \sin nx$$

$$= \frac{2E}{\beta(\pi - \beta)}\left[\sin\beta \sin x - \frac{1}{4}\sin 2\beta \sin 2x + \frac{1}{9}\sin 3\beta \sin 3x - \cdots\right]$$

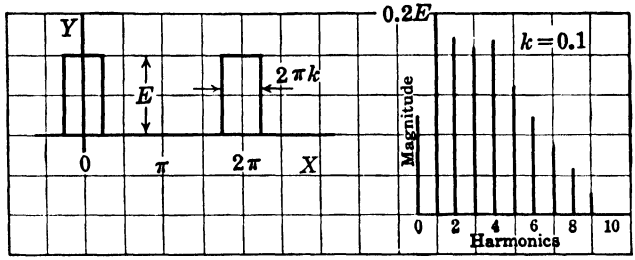
Scanning wave with β flyback time:



$$y = \frac{2E}{\pi\beta} \sum_{n=1}^{n=\infty} \frac{(-1)^{n+1}}{n^2} \sin n\beta \sin nx$$

$$= \frac{2E}{\pi\beta}\left[\sin\beta \sin x - \frac{1}{4}\sin 2\beta \sin 2x + \frac{1}{9}\sin 3\beta \sin 3x - \cdots\right]$$

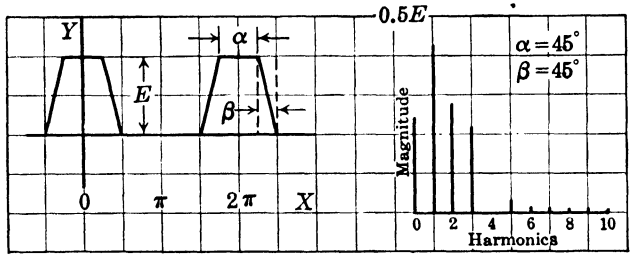
***k.* Rectangular pulse wave:**



$$y = kE + \frac{2E}{\pi} \sum_{n=1}^{n=\infty} \frac{1}{n} \sin nk\pi \cos n(x)$$

$$= kE + \frac{2E}{\pi} \left[\sin k\pi \cos (x) + \frac{1}{2} \sin 2k\pi \cos 2(x) + \cdots \right]$$

***l.* Symmetrical trapezoidal pulse wave:**

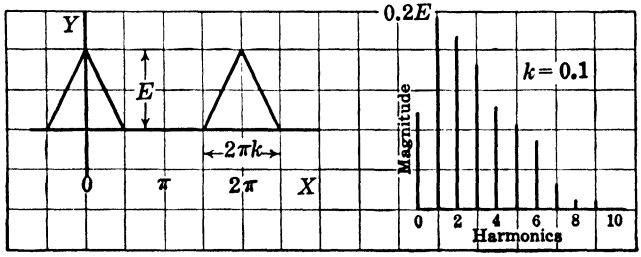


$$y = \frac{E}{2\pi} (\alpha + \beta) + \frac{8}{\beta} \sum_{n=1}^{n=\infty} \frac{1}{n^2} \sin \frac{n\beta}{2} \sin \frac{n}{2} (\alpha + \beta) \cos nx$$

$$= \frac{E(\alpha + \beta)}{2\pi} + \frac{4E}{\pi\beta} \left[\sin \frac{\beta}{2} \sin \left(\frac{\alpha + \beta}{2} \right) \cos x \right.$$

$$\left. + \frac{1}{4} \sin \beta \sin (\alpha + \beta) \cos 2x + \cdots \right]$$

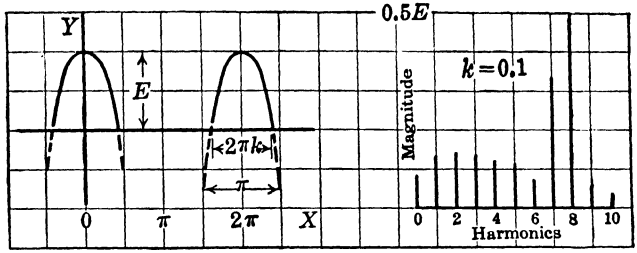
***m.* Symmetrical triangular pulse wave:**



$$y = kE + \frac{2E}{k\pi^2} \sum_{n=1}^{n=\infty} \frac{1}{n^2} (\sin nk\pi)^2 \cos nx$$

$$= kE + \frac{2E}{k\pi^2} \left[\sin^2 k\pi \cos x + \frac{1}{4} \sin^2 2k\pi \cos 2x + \cdots \right]$$

n. **Fractional cosine pulse wave:**



$$y = \frac{E}{\pi(1 - \cos k\pi)}\left[\sin k\pi - k\pi\cos k\pi + (k\pi - \sin k\pi\cos k\pi)\cos x + \sum_{n=2}^{n=\infty}\left\{\frac{\sin(n+1)k\pi}{n+1} + \frac{\sin(n-1)k\pi}{n-1} - \frac{2\sin nk\pi\cos k\pi}{n}\right\}\cos nx\right]$$

$$= \frac{E}{\pi(1 - \cos k\pi)}\left[\sin k\pi - k\pi\cos\pi k + (k\pi - \sin k\pi\cos k\pi)\cos x + \left(\frac{1}{3}\sin 3k\pi + \sin k\pi - \sin 2k\pi\cos k\pi\right)\cos 2x + \left(\frac{1}{4}\sin 4k\pi + \frac{1}{2}\sin 2k\pi - \frac{2}{3}\sin 3k\pi\cos k\pi\right)\cos 3x + \cdots\right]$$

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ANSWERS TO EXERCISES

Chapter 1

1.1. (*a*) All elements have in common the property of being able to exist in three states; solid, liquid, and gas. Another property all elements have in common is their ability to combine with other elements to form compounds. (*b*) All atoms are alike in that they are composed of positive protons and negative electrons.

1.2. Orbital electrons revolve about the nucleus of the atom. They are related to the atomic weight in that the number of orbital electrons is approximately twice the atomic weight. There is some variation in the atomic weight owing to the occurrence of isotopes.

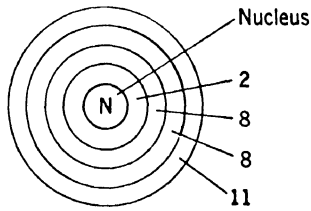
1.3. A good conductor has many free electrons in a given volume, while an insulator has very few free electrons in the same volume.

1.4. When a substance capable of forming ions is immersed in a dissolving solution, it will divide into positive and negative ions.

1.5. A stable ion results when an ion of an active element attracts or releases an electron, thereby becoming electrically charged but losing its ability to combine actively with other elements. Because of its inactive qualities in this charged condition, it is called a "stable" ion.

1.6. When an atom having an orbital electron is placed between two oppositely charged plates, the path of the electron will be distorted, the electron being attracted toward the positive plate and the positive nucleus being attracted toward the negative plate.

1.7. The relative positions of the electrons in a copper atom are as shown in this illustration.



1.8. The chemical and physical properties of an element are determined by the amount of positive charge on the nucleus, weight of nucleus, and number of orbital electrons.

1.9. $m = 9.1 \times 10^{-28}$ g.

1.10. $U = 8.19 \times 10^{-7}$ erg.

Chapter 2

2.1. 4.445×10^{19} electrons.

2.2. 44.72 sec.

2.3. 28.06 amp.

2.4. Using platinum (+0.87) and lead (−0.13) will give a difference of potential of 1 volt.

2.5.
$$PbO_2 + Pb + 2H_2SO_4 \underset{\text{Discharging}}{\overset{\text{Charging}}{\rightleftharpoons}} 2PbSO_4 + 2H_2O$$

2.6. $l = 940$ ft.

2.7. $\alpha_1 = 0.003$. The metal is likely to be platinum.

2.8. $W = 0.02$ joules.

2.9. $I = 0.1825$ amp.

2.10. Delay action in fuses is obtained by making very short thin strips of fuse material adjacent to large areas that will absorb the heat away for a short time and delay the blowing of the fuse.

Chapter 3

3.1. $R_x = 3.0111$ ohms, $V = 27.1$ volts.

3.2. $I = 2.5$ amp.
$I_2 = 0.5$ amp.
$R_1 = 55$ ohms.

3.3. $I_1 = 69.514$ amp.
$I_2 = 10.972$ amp.
$I_3 = 21.944$ amp.
$I_4 = 36.573$ amp.
$G_2 = 2$ mhos.
$G_3 = 4$ mhos.
$G_4 = 6.666 \ldots$ mhos.

3.4. (*a*) $R_s = 0.006001$ ohm; (*b*) $R_m = 49{,}970$ ohms.

3.5. $I_1 = 0.38$ amp $= 380$ ma.
$R_1 = 184.21$ ohms or 26.57 watts. Use 30 watts.
$R_2 = 500$ ohms or 16.2 watts. Use 20 watts.
$R_3 = 450$ ohms or 4.5 watts. Use 5 watts.

Chapter 4

4.1. $E_{cb} = 450$ volts; $E_{ca} = 562.5$ volts.

4.2. $R_{eq} = 90$ ohms; $E_{gh} = 0.69$ mv actual output voltage.

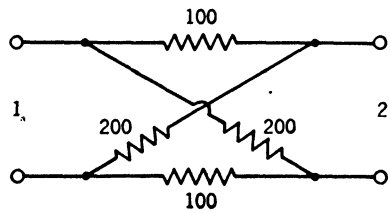
Note: An error in the first part of this problem may cause a large error in the final answer.

4.3. $R_a = 246.66 \ldots$ ohms; $R_b = 296$ ohms; $R_c = 370$ ohms.

4.4. $R_1 = 290.909 \ldots$ ohms; $R_2 = 363.636 \ldots$ ohms; $R_3 = 727.272 \ldots$ ohms.

4.5. $R_1 = 37$ ohms; $R_2 = 55$ ohms; $R_3 = 1.6$ ohms.

4.6. $R_1 = R_2 = 50$ ohms; $R_3 = 100$ ohms.



Lattice network with values.

4.7. $I_1 = -0.375$ amp; $I_2 = 1.25$ amp; $I_3 = 0.875$ amp.

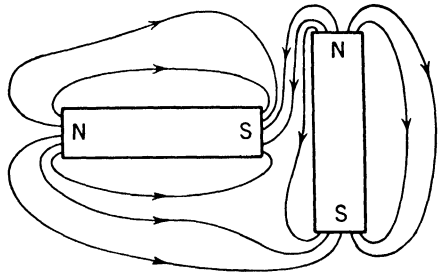
4.8. $R_3 = 960$ ohms, $R_4 = 540$ ohms.

4.9. $R_x = 149.24$ ohms.

4.10. $X = \dfrac{R_4 R_L}{r_2(R_3 + R_4)}$

Chapter 5

5.1.



Sketch of magnetic field and direction of lines of force around two permanent magnets.

5.2. The magnetic field about a straight conductor carrying current in free space is in the form of concentric circles about the conductor. The magnetic field is in the same direction as the fingers when the conductor is clasped by the right hand and the thumb points in the direction of current flow.

5.3. $H = 26.5$ oersteds.

5.4. $F = 188.5$ gilberts.

5.5. $R_{\text{total}} = 0.003475$ rel in cgs units; $F = 69.5$ gilberts.

5.6. $R_{\text{total}} = 0.1478$ rel in cgs units; $F = 2{,}955.9$ gilberts. With air gap correction for fringing, $R_{\text{total}} = 0.1263$ rel in cgs units; $F = 2{,}526$ gilberts.

5.7. $R_a = 0.001575$ rel in cgs units; $\Phi_a = 134{,}400$ maxwells.
$R_b = 0.0003937$ rel in cgs units; $\Phi_b = 100{,}800$ maxwells.
$R_c = 0.001181$ rel in cgs units; $\Phi_c = 33{,}600$ maxwells.

5.8. $l = 0.0424$ in. = length of air gap (neglecting fringing); $\Phi_b = 20{,}543$ maxwells; $\Phi_c = 79{,}457$ maxwells.

5.9. $l = 0.0484$ in. = length of air gap (accounting for fringing).

5.10. $\alpha_{\text{iron}} = 3°16'$ = angle the flux line makes with the surface in the cast iron.

Chapter 6

6.1. 0.01 volt.

6.2. No.

6.3. 0.1 volt.

6.4. (*a*) In a series-wound generator, the armature and field windings are in series. (*b*) In a shunt-wound generator, the armature and field windings are in parallel.

6.5. In a dynamotor the motor and generator windings are on the same armature, while in a motor generator these windings are on separate armatures.

6.6. This core serves to concentrate the magnetic field of the pole pieces and makes it uniform.

6.7. Winding the coil on a conducting frame damps the movement.

6.8. Dynamometer, moving-iron or moving-vane type of meter, thermocouple type of meter, and rectifier type of meter.

6.9. By the amount of current required to operate it.

6.10. A circuit breaker is a device for opening a circuit when the current in the circuit exceeds a certain value.

Chapter 7

7.1. In an inductive circuit, the generated voltage always opposes any change in the current through the circuit.

7.2. Inductance is analogous to mass.

7.3. The unit of self-inductance is the henry.

7.4. $e = -L\dfrac{di}{dt}$

7.5. When flux lines generated in one coil cut the turns of another coil, the two are said to be "coupled by mutual inductance."

7.6. The time constant of an inductive circuit is the time required for the current to rise 63.2 per cent of its final value after a voltage is applied. It equals L/R.

7.7. 6.32 amp.

7.8. 10 henrys.

7.9. $L = \dfrac{a^2n^2}{9a + 10b}$

7.10. $L = \dfrac{(4)(100)}{18 + 20} = 10.5\ \mu\text{h}$

Chapter 8

8.1. When a capacitor is charged, the plates assume opposite potentials. Static lines of force start at the positive and terminate on the negative plate. Since any electrostatic field contains energy, the capacitor is said to have a charge of q coulombs, which depends directly upon the capacity of the capacitor C and the applied voltage E; that is, $Q = CE$. Physically the dielectric in the capacitor experiences stresses due to the electrostatic field. The atoms of the dielectric are distorted so that the electron orbits are shifted toward the positive plate. The amount of this elasticity in the atoms of the dielectric determines the dielectric constant of the material.

8.2. The force between two charged bodies depends directly upon the product of the charges on the bodies and inversely upon the square of the distance between them and the dielectric constant of the medium. This is known as "Coulomb's law."

8.3. The electrostatic potential between two points can be defined as the work required to move a unit positive charge from one point to the other. If the charge is moved in opposition to the electrostatic force, the charge obtains potential energy. The field intensity through which the charge is moved may be expressed in volts per inch.

8.4. $Q = 0.01$ coulomb.

8.5. When a voltage is applied to a capacitor, a current will flow. This current will consist of a charging current, which builds up a charge on the capacitor plates and establishes the electrostatic field, displacing the electronic orbits in the dielectric, and a leakage current, which will continue to flow owing to the imperfectness of the dielectric. The charging current can be only a variable current during charge and ceases to flow when the capacitor is fully charged.

8.6. Normally it will take 450,000 volts to break down a 6-in. air gap.

8.7. Corona is caused by high voltages, usually at sharp corners of the conductor, where a discharge takes place, and may appear visibly as a glow and audibly as

a hiss. Corona effects represent a power loss, hence, should be avoided by making large smooth curves in conductors and avoiding sharp corners on high-voltage electrical equipment.

8.8. 0.1897 μf/mile.

Chapter 9

9.1. If a loop of conductor is rotated at a uniform angular velocity in a uniform magnetic field, the flux linkage through the loop of conductor will form a sinusoidal curve if plotted against time as shown in Fig. 9.2. The voltage induced in the conductor will be proportional to the negative slope of the flux curve; hence the voltage induced in the conductor making a sinusoidal flux linkage as shown in Fig. 9.2 will be like the sinusoidal induced-voltage curve shown in Fig. 9.3.

9.2. Phase angle is the angle between two rotating (maximum) vectors, which may be current and voltage vectors. It is usually expressed as an angle but can be expressed as the time between positive maximum values. It is valuable in explaining and calculating a-c circuits.

9.3. (*a*) This curve is similar to that shown in Fig. 9.4. Its equation is

$$\Phi = 1.1 \times 10^{10} \cos \theta$$

(*b*) $\Phi = 8.43 \times 10^9$ at 40°, 3.76×10^9 at 70°, 0 at 270°, 1.083×10^{10} at 350°.

9.4. (*a*) $E_m = 2.073 \times 10^6$ = maximum voltage.
(*b*) $E = 1.466 \times 10^6$ = effective voltage.
(*c*) $e = 2.073 \times 10^6 \sin \omega t$ = instantaneous voltage.
(*d*) Since $f = 60$ cycles/sec, the instantaneous voltage at the end of 2 sec will be zero.

9.5. Crest factor is the ratio of the maximum value to the effective value, and the *peak factor* is the same as crest factor. *Effective value* = root mean square, which means the square root of the sum of the squares. *Average value* is the average of the instantaneous values over a half cycle. *Maximum value* is the largest instantaneous value during a cycle. *Form factor* is the ratio of effective value to average value.

9.6. $I_m = 5.66$ amp = maximum a-c current; $I = 4.0$ amp effective a-c current. 4.0 effective amp in an a-c circuit will cause the same heating as 4.0 d-c amp; hence the maximum value of alternating current is (1.414)(4) = 5.66 amp. The average value of alternating current is

$$I_{av} = (0.637)(5.66) = 3.605 \text{ amp}$$

9.7. 218.5 watts.

9.8. 24.3 scale divisions.

9.9. Curves are similar to those of Fig. 9.5 of this chapter. E_{m1} and E_{m2} are not equal. The phase angle E_{mt} makes with E_{m1} is 51°21′.

Chapter 10

10.1. Electrons are negative charges of electricity having electrostatic lines of force terminating on them. If the electron is set into motion, the electrostatic lines of force move along with the electron, creating an electromagnetic field.

10.2. $R = 10$ ohms effective resistance. Such a vector diagram is shown in Fig. 10.1*b*.

10.3. $X_L = 4{,}666.6$ ohms inductive reactance. Such a vector diagram is shown in Fig. 10.2*b*.

10.4. $X_C = 4{,}500$ ohms capacitive reactance. See Fig. 10.3*b* for vector diagram of a pure capacitive reactance.

10.5. (*a*) $L = \dfrac{X_L}{2\pi f}$

(*b*) $C = \dfrac{1}{2\pi f X_C}$

10.6. (*a*) $L = 50$ mh.

(*b*) $\omega = 246.9$ radians/sec; $f = 39.3$ cycles/sec.

10.7. 128.6 mh.

10.8. 116.8 μf capacity.

10.9. (*a*) $X_L = 14{,}700$ ohms.

(*b*) $f = 12.38$ Mc.

(*c*) $L = 31.8$ mh.

10.10. (*a*) $X_C = 204.1$ ohms.

(*b*) $f = 126.3$ cycles/sec.

(*c*) $C = 400.5$ $\mu\mu$f.

Chapter 11

11.1. *Average power* can be defined as the area under the instantaneous-power curve divided by the time, usually plotted as the abscissa. This gives the average ordinate. *Phase angle*, usually expressed by ϕ, is the angle between the voltage and current vectors. *Power factor* is the ratio of the real power, of a pure sine wave, divided by the apparent power and can be expressed by $\cos \phi$. *Real power* is equal to the in-phase component of voltage multiplied by the current. *Apparent power* is equal to the product of the voltage and the current, regardless of their phase relationship. *Reactive factor* is the ratio of the reactive power to the apparent power and can be expressed by $\sin \phi$.

11.2. Alternating-current machinery is rated on the basis of apparent power because eddy currents and hysteresis losses are a function of the magnetic flux, and hence the voltage generated and the resistance or copper loss in the winding is a function of the current that flows. Ratings are based upon temperature rise; hence, apparent power, which is the product of the voltage and the current regardless of phase angle, is the best measure of this loss in the machinery.

11.3. (*a*) $e = 250\sqrt{2} \sin \omega t = 353.5 \sin \omega t \quad \mu$v

$i = \dfrac{\sqrt{2}}{2} \sin \omega t = 0.707 \sin \omega t \quad \mu$a

$p = 125 - 125 \cos 2\omega t \quad \mu\mu$w

(*b*) $2\pi f = 2(5{,}000)$ for the power wave

or $f = 1{,}592$ cycles/sec.

11.4. (*a*) $Z = 500$ ohms.

Phase angle $\phi = 0°$.

Reactive factor $= 0$.

(*b*) $I = 0.5\ \mu a$ = effective current

$E = IZ = IR = 250\ \mu v$ = effective voltage $\longrightarrow$ maximum voltage vector (100 μv/in.)

(*c*) $P = 125\ \mu\mu w$ = real power

11.5. The load of Exercise 11.3 never feeds energy back into the line, because the load is resistive and the product of EI is never negative, a necessary condition to feed energy back into the line.

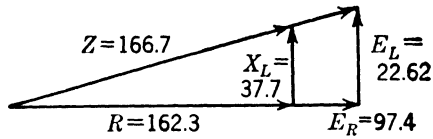
11.6. E = 1,355 volts effective.
E_m = 1,916 volts maximum.
I = 36.9 amp effective.
I_m = 52.2 amp maximum.

11.7. (*a*) R = 162.3 ohms.
(*b*) $\cos \phi = 0.974$ = the load power factor.

11.8. The scales are

$$\begin{matrix} 2 \text{ ohms} = 1 \text{ mm}, \\ 1 \text{ volt} = 1 \text{ mm} \end{matrix} \qquad E = 100$$

$E_L = 22.62$ = the voltage across the inductance.
$E_R = 97.4$ = the voltage across the resistance.



Vector diagram of impedance and voltage.

11.9. $E_C = 132.6$ = the capacitor voltage.
$E = 132.9$ = the voltage across the two elements in series.

11.10. Z = 332.6 ohms = impedance.
$P = EI = (132.9)(0.4) = 53.2$ watts = apparent power.
$\cos \phi = 0.0752$ = the power factor.
$\sin \phi = 0.9972$ = the reactive factor.

Chapter 12

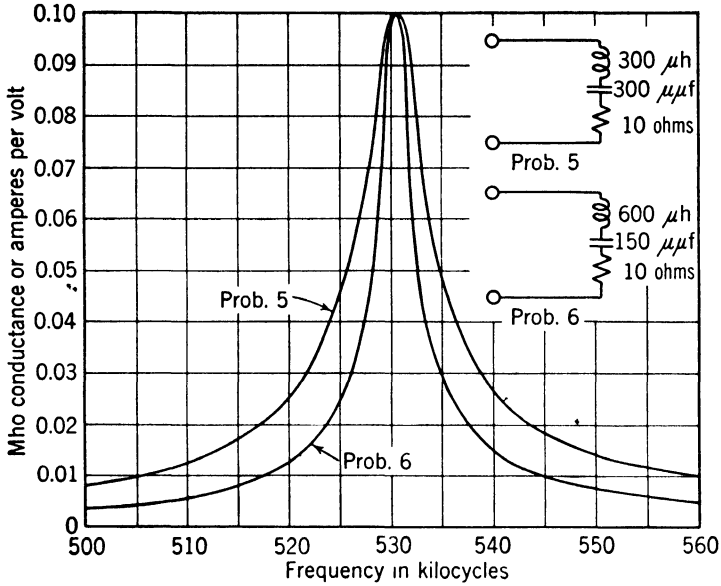
12.1. The factor $Q = \omega L/R$ determines the characteristics of a series circuit. If Q is increased, the resonance curves increase in height and sharpness because the resistance decreases and hence the current is not limited so much at resonance.

12.2. f_r = 530.5 kc.
Z = 10 ohms impedance.
$\phi = 0°$. (The voltage and current are in phase.)

12.3. 536 kc and 525 kc.

12.4. I_r = 25 amp current at resonance.
I = 17.68 amp when $\cos \phi = 0.707$.
E_L = 17,769 or 17,591 volts.
E_C = 17,591 or 17,769 volts.

12.5. See illustration.



12.6. The two circuits shown in the illustration have the same resonant frequency because the LC product is the same in either case. It will be noted that the curves are not identical, since the Q is not the same in the two cases even though the resistances of the circuits are identical.

12.7. The LC product determines the resonant frequency, and the L/C ratio determines the selectivity of the circuit, which is limited by mechanical design and the Q of the circuit.

12.8. Yes, because there can be positive and negative reactances, which cancel each other when viewed from the input terminals. However, current flowing through these elements must develop voltages across them proportional to the value of their respective reactances. Exercise 12.4 shows how high voltages can be developed across such a circuit.

12.9. $Z_1 = 100\underline{/+60°} = 100(\cos 60° + j \sin 60°)$

$= 100(0.50 + j0.866) = 50 + j86.6$

$Z_2 = 75\underline{/-30°} = 75(\cos 30° - j \sin 30°)$

$= 75(0.866 - j0.50) = 64.95 - j37.5$

$Z_3 = 70.7\underline{/-45°} = 50 - j50$

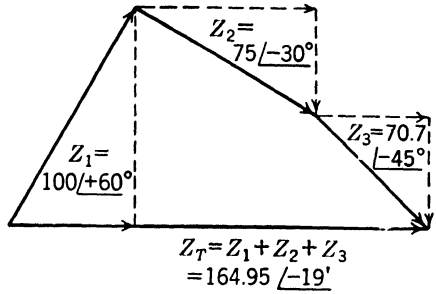
(*a*) $Z_T = Z_1 + Z_2 + Z_3 = 164.95\underline{/-19'} = 164.95 - j0.9$ ohms = total impedance of the circuit. See illustration on the next page.

(*b*) $I = \dfrac{E}{Z} = \dfrac{500/0°}{164.95\underline{/-19'}} = 3.031\underline{/+19'}$ amp = the current with its phase relation to the applied voltage.

(*c*) $|E_1| = I|Z_1| = 303.1$ volts across Z_1

$|E_2| = I|Z_2| = 227.4$ volts across Z_2

$|E_3| = I|Z_3| = 214.3$ volts across Z_3



Vector diagram for Exercise 12.9.

Chapter 13

13.1. The reactance curves become less sharp as the Q of a parallel circuit decreases. The same is true of the resistance and impedance curves. Figures 13.2 and 13.8 illustrate this point.

13.2. (*a*) $f_r = 1.0$ Mc.

(*b*) $Q = 47.15$ and $\frac{1}{2Q} = \frac{1}{94.3}$.

$\frac{f_r}{2Q} = 10{,}600$ = the difference between actual and resonant frequency.

Then $X_L = 466.8$ and $X_C = 466.2$. These are the maximum inductive and capacitive reactances.

(*c*) $f_L = 989{,}400$

$f_C = 1{,}010{,}600$

13.3. $Z = \frac{L}{RC} = 22{,}190$ ohms.

$Z = 44{,}380$ ohms when $R = 5$ ohms.

13.4. $C = 168.8$ μf.

Yes, because the Q is higher.

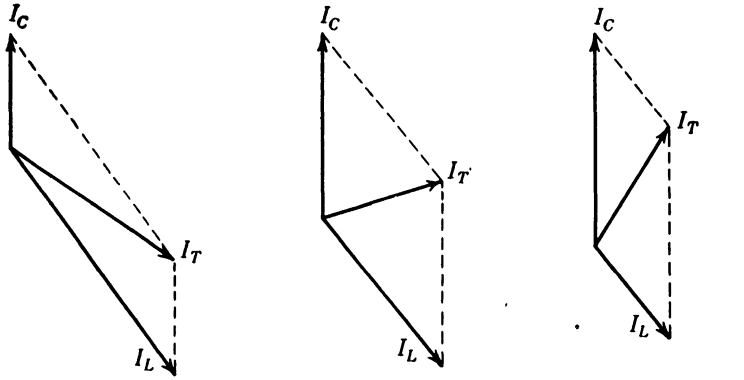
13.5. A meter can be used in the line to a parallel circuit to determine antiresonance by obtaining a minimum reading. When antiresonance occurs, the parallel circuit looks like a very large impedance; hence minimum current will flow through the meter.

13.6. At antiresonance a parallel circuit is resistive. For lower frequencies, the circuit is inductive, and for higher frequencies it is capacitive. At antiresonance $X_L = X_C$, and the circuit becomes resistive because the effects of these two elements in parallel cancel. Below resonance X_C is greater than X_L and hence the circuit looks inductive, or like the lower valued reactance; above resonance, X_C is smallest and hence the circuit looks capacitive.

13.7. $R = \sqrt{L/C} = 22.36$ ohms in each branch.
$Z = 22.36$ ohms.

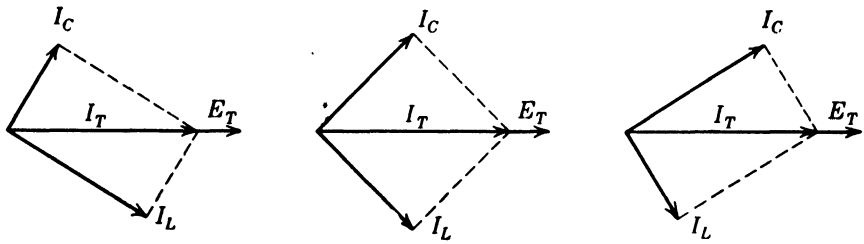
13.8. Susceptance curves offer a convenient method of adding the effect of impedance elements connected in parallel. Complex circuits can be analyzed qualitatively by the use of this tool to determine the number of resonant and antiresonant frequencies. The total susceptance changes from negative to positive as the frequency is increased through antiresonance.

13.9. See illustration. At antiresonance the capacitive current is equal to the inductive current. Below antiresonance the inductive current becomes larger, and above antiresonance the capacitive current becomes larger.



Below antiresonance | At antiresonance | Above antiresonance

13.10. Drawing current vectors in milliamperes for 1 volt to a scale of 1 ma = 1 mm, we have the vector diagrams shown in the illustration. The terminal points of these vectors traverse a circle. The sum of the currents I_C and I_L is I_T, which is always in phase with E_T; hence the circuit has unity power factor and always looks like a pure resistance.



$f_1 = 1643$
$I_L = 38.7/-30°$
$I_C = 22.38/+60°$
$I_T = 44.7/0°$

$f_2 = 2850$
$I_L = 31.63/-45°$
$I_C = 31.63/+45°$
$I_T = 44.7/0°$

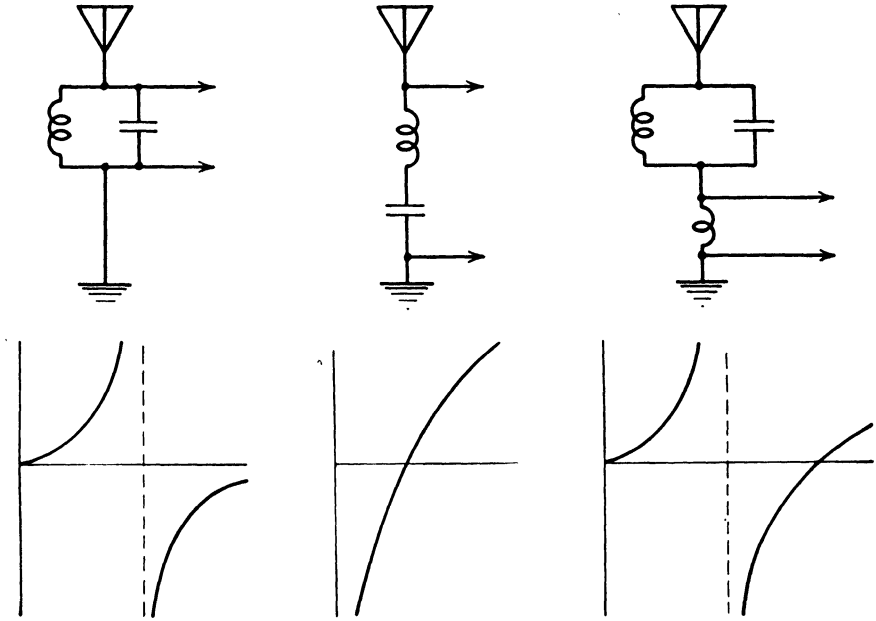
$f_3 = 4934$
$I_L = 22.38/-60°$
$I_C = 38.7/+30°$
$I_T = 44.7/0°$

Chapter 14

14.1. A wave trap is a combination of inductance and capacity arranged to utilize the principle of resonance to eliminate undesired signals. The wave trap is arranged to offer a high impedance to the undesired signal and a low impedance

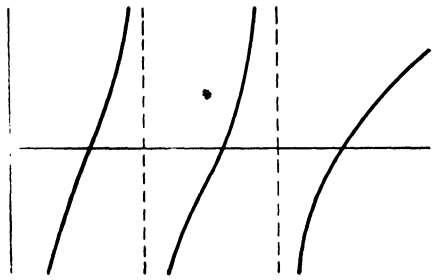
to the desired signal. This can be accomplished in either series circuits, parallel circuits, or combinations of these circuits.

14.2. See the illustration.



Wave trap filter characteristics.

14.3. This circuit has three resonant points and two antiresonant points, as shown in the diagram.



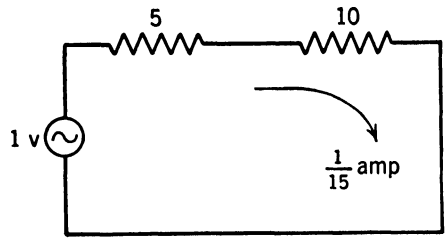
Filter reactance curves.

14.4. $X = 15$ ohms capacitive reactance.

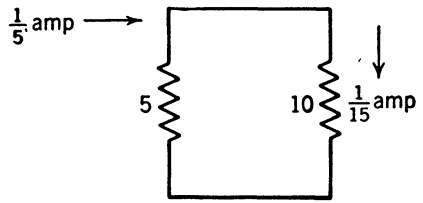
14.5. $Z = 129 + j276$ ohms impedance.

Chapter 15

15.1. See illustration.



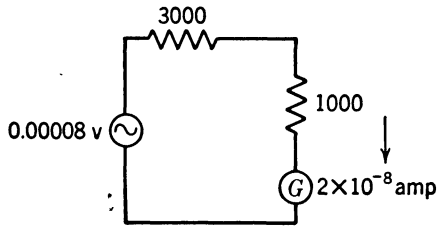
15.2. See illustration.



15.3. $\frac{1}{15}$ amp.

15.4. $\frac{2}{15}$ amp.

15.5. See illustration.



15.6. 50 kw.

15.7. $10 - j0.8$ ohms.

15.8. $7.09 + j7.09$ ohms.

15.9. 10 amp.

15.10. The circuit elements must be linear.

Chapter 16

16.1. The Edison effect is the phenomenon of a weak current flowing to a plate inside an evacuated tube when the plate is connected to the positive end of the filament.

16.2. $I = \frac{P}{E} = \frac{100}{100} = 1$ amp $= 6.285 \times 10^{18}$ electrons/sec

$= 1.982 \times 10^{26}$ electrons/yr.

Weight of electrons in 1 yr = 0.1784 g.

$\frac{0.02}{0.1784} = 0.1121$ yr or $(365)(0.1121) = 40.88$ days.

16.3. $F = 1.327 \times 10^{-9}$ dynes.
$v = 9.405 \times 10^{8}$ cm/sec $= 5{,}840$ miles/sec.
$t = 6.38 \times 10^{-10}$ sec.

16.4. $\bar{v} = \frac{v}{2} = 4.702 \times 10^{8}$ cm/sec.
$n = 8.4 \times 10^{28}$ electrons/cm.

16.5. The work function of a metal is the energy lost by a unit charge when passing through the surface of the metal. Thermionic emission increases as the work function decreases.

16.6. For example, if the thoriated-tungsten emitter is operated above temperature, the evaporation of pure thorium will be excessive and will shorten the tube life; if this emitter is operated below temperature, the emission will decrease owing to the fact that the evaporated atomic layer is not replenished rapidly enough from the supply of pure thorium within the emitter. If the tube is operated properly, the evaporation of the atomic surface will be replenished by pure thorium from within the emitter without excessive evaporation.

16.7. Thoriated-tungsten emitters can be rejuvenated by *flashing* at a high temperature to reduce more thorium oxide to pure thorium and by *aging* at a high temperature to permit a supply of pure thorium to form an atomic layer at the surface.

16.8. Pure-tungsten emitters are constructed by working the tungsten at a high temperature into a fibrous condition so that it can be formed into the proper shape for the filament. The filament has such a high melting point and is made of such heavy atoms that very good emission properties are obtained in spite of the relatively large work function of the metal. The power consumption for tungsten filaments is quite large, owing to the large size and high operating temperature of the filament. The high operating temperature causes considerable atomic evaporation, which definitely shortens the filament life before it burns out. Since tungsten has such a high melting point, high evacuation at high temperature is practical. Furthermore, tungsten has an affinity for gas and thus maintains a high vacuum, which is essential in high-power tubes operating at high voltages. Tungsten-filament tubes find most use in the field of high-power vacuum tubes.

16.9. Carbonized thoriated-tungsten emitters are made by operating a thoriated-tungsten emitter in an atmosphere of hydrocarbon to form a surface layer of tungsten carbide. This treatment reduces atomic thorium evaporation to 15 per cent of the original value and permits higher operating temperatures. This type of emitter has the advantage of high emission and high operating temperature; thus it competes with pure-tungsten-filament tubes for high-power operation.

16.10. Owing to the unequal potential along the filament, the greatest amount of emission takes place at the negative end, because the filament-to-plate voltage is greatest at this end. Since all the space current flows in at the negative terminal (or effectively increases the current at the negative end in comparison to the positive end by that amount), the temperature of the negative end of the filament will be increased to increase the emission at this end. The increased emission at the negative end will increase the filament temperature as a result of the energy liberated when electrons overcome the work function at the surface of the filament.

Chapter 17

17.1. Unilateral conductivity is a low resistance in one direction of current flow and infinite resistance when current tries to flow in the opposite direction. Crystal rectifiers and vacuum tubes are good examples of devices having low resistance in one direction and very high resistance in the other direction.

17.2. The total emission current is $I_S = (0.175)(40.5) = 7.09$ ma.

17.3. The emission of electrons causes a negative charge between the plate and cathode when the plate is connected to the cathode. This is shown in Fig. 17.2*b*.

17.4. The plate current varies as the $\frac{3}{2}$ power of the plate voltage in all types of tubes. Child first developed this relation. It is assumed that the electrons leave the cathode at zero initial velocity.

17.5. $I_b = 110.7$ ma.

17.6. When the plate voltage is increased, more electrons drift to the plate to diminish the negative space charge. The remaining space-charge crest is smaller and moved closer to the cathode.

17.7. Since diode emission depends upon cathode temperature and plate voltage, the three variables can be represented in three dimensions. To see the complete picture, it is convenient to view the diode's characteristics as shown in Fig. 17.9.

17.8. The d-c resistance will be equal to the a-c resistance only when the tangent to the curve passes through the origin. This is a special case that occurs at only one point on the curve (at approximately $E_b = 22.5$ in Fig. 17.10). The d-c resistance is different from the a-c resistance because in general the slope of the curve does not pass through the origin.

17.9. $r_{b1} = 3{,}077$ ohms.
$r_{b2} = 3{,}235$ ohms.

17.10. $r_p = 6{,}667$ ohms.

Chapter 18

18.1. A triode is an evacuated enclosure containing a cathode, grid, and anode. For a free grid the plate resistance is the same as for a diode. If the grid is made more positive, the plate resistance decreases, and if the grid is made more negative, the plate resistance increases.

18.2. The lower curvature is due to space-charge effect resulting from a large supply of electrons being emitted from the cathode with initial velocities. The upper curvature is due to plate voltage saturation, resulting from a lack of electrons available for the plate.

18.3. The three families of grid characteristic curves are valuable for graphical solution of problems involving positive-grid operation. It is considerably more difficult to approximate these curves mathematically because of their complex nature. The grid resistance is infinite for a negative grid and decreases as the grid is made more positive.

18.4. The static characteristics are in error at high frequencies when electron transit time is appreciable. The tube constants μ, R_p, and g_m can be used in mathematical expressions to approximate characteristics involving plate current.

18.5. From Fig. 18.3 and Eq. (18.1) we can write

$$\mu = -\frac{dE_b}{dE_c} \doteq -\frac{E_{b1} - E_{b2}}{E_{c1} - E_{c2}} = -\frac{50 - 25}{10 - 20} = 2.5$$

From Fig. 18.5 and Eq. (18.2), when $E_c = +10$, we can write

$$r_p = \frac{dE_b}{dI_b} \doteq \frac{E_{b1} - E_{b2}}{I_{b1} - I_{b2}} = \frac{25 - 50}{0.01 - 0.02} = 2{,}500 \text{ ohms}$$

From Eq. (18.6)

$$g_m = \frac{\mu}{r_p} = \frac{2.5}{2{,}500} = 1{,}000 \ \mu\text{mho}$$

18.6. By Eq. (18.6) $g_m = 5{,}250$ μmho
By Eq. (18.10) for no load resistance

$$4000I_b - 21E_c - 2000 = 0$$

and for a load resistance

$$16{,}500I_b - 21E_c - 2000 = 0$$

18.7. By Eq. (18.8) the instantaneous grid voltage is

$$e_c = -45 + 42.4 \sin \omega t$$

By Eq. (18.17) the instantaneous plate voltage is

$$e_p = \frac{(400)(800) - (4.2)(-45)(2{,}500)}{800 + 2{,}500} - \frac{(4.2)(42.4)(2{,}500)}{800 + 2{,}500} \sin \omega t$$

$$= \frac{320{,}000 + 472{,}500}{3{,}300} - \frac{445{,}000}{3{,}300} \sin \omega t$$

$$= 240 - 135 \sin \omega t$$

18.8. By Eq. (18.14)

$$i_b = \frac{(4.2)(-45) + 400}{800 + 2{,}500} + \frac{(4.2)(42.4)}{800 + 2{,}500} \sin \omega t$$

$$= \frac{211}{3{,}300} + \frac{177}{3{,}300} \sin \omega t$$

$$= 63.9 + 53.9 \sin \omega t \qquad \text{(in milliamperes)}$$

18.9. By Eq. (18.20)

$$I_p = \frac{(4.2)(30)}{800 + 2{,}500} = 38.2 \text{ ma}$$

By Eq. (18.22)

$$A = \frac{-(4.2)(2{,}500)}{800 + 2{,}500} = -3.18 = \text{useful amplification}$$

18.10. $P_{ac} = I_p^2 r_p + I_p^2 R_L = (0.0382)^2(800) + (0.0382)^2(2{,}500)$
$= 1.17 + 3.65 = 4.82$ watts
$P_{dc} = I_b^2 r_p + I_b^2 R_L = (63.6)^2(800) + (63.6)^2(2{,}500) = \text{mw}$
$= 3.24 + 10.11 = 13.35$ watts
The plate voltage could be lowered from 400 to 240 volts to give the same results, and the d-c plate power saved would be 10.11 watts.

Chapter 19

19.1. Prototype amplifier = triode.

The derived types of amplifier with their contributions are:

Tetrode. Shields input from output to make the tube truly unilateral as well as increase its sensitivity.

Pentode. The suppressor grid controls secondary emission to increase the efficiency and plate-voltage swing.

Beam-power pentode. Replaces the suppr ssor grid with a virtual suppressor grid and concentrates the electrons to beams to increase still further the efficiency and plate-voltage swing.

19.2. Since the mutual conductance is the incremental change of electron plate current divided by a corresponding incremental change in grid voltage, we would expect no essential difference of mutual conductance in the triode, tetrode, and pentode because the effect of the control grid on the electron plate current is essentially the same in these tubes.

19.3. In a space-charge tetrode, the first grid is operated at a positive potential to eliminate the space charge, which ordinarily would prevent the plate current from having a sharp cutoff.

19.4. The primary difference between the static plate characteristics of a pentode as compared to those of a tetrode is that in the pentode, secondary emission from the plate is eliminated to make the tube stable over the whole range of plate voltages. This increases the useful operating range of plate voltage. The shape of ideal static plate characteristics would be straight parallel equidistant lines for all values of plate voltage.

19.5. A hexode tube containing four grids could be used as follows:

Grid 1. Space-charge grid, small positive voltage
Grid 2. Control grid, small negative voltage
Grid 3. Screen grid, large positive voltage
Grid 4. Suppressor grid, connected to cathode

This type of tube should have a lower plate resistance than the pentode and a sharper cutoff of plate current.

19.6. The power output of pentodes is limited primarily by the curvature of the plate characteristics at low plate voltages. The beam-power pentode with its virtual suppressor grid does not reject any primary electrons in this region so long as the minimum potential in this region exceeds zero. The transition from the region of the static plate characteristics where the plate current is substantially independent of plate voltage and where the plate current is primarily a function of the plate voltage is very abrupt and takes place at a very low plate voltage. The variable-μ action of a physical suppressor grid is eliminated by using the virtual grid of a beam-power pentode.

19.7. Power sensitivity $= \dfrac{\sqrt{\text{output a-c power}}}{\text{input a-c voltage}}$

The screen-grid current in a beam-power pentode is made low by placing the screen wires physically in the electron shadow of the control-grid wires; thus very few electrons find their way to the screen to form screen-grid current.

19.8. Equation (19.4) is for a constant-μ tube; hence it cannot be used for a variable-μ pentode.

19.9. Several useful applications of the pentagrid-mixer amplifier are: (1) as high-gain pentode amplifier, (2) as compressor amplifier, (3) as expander amplifier, and (4) for mixing two signals.

19.10. The justification of multifunction tubes is to simplify circuits, reduce costs, and save space.

Chapter 20

20.1. The electron gun supplies a beam of high-speed electrons.

20.2. The electron beam becomes visible when it strikes the phosphorescent screen.

20.3. The intensity of the beam is varied by varying the potential of the grid or the potential of the accelerating anode.

20.4. By coating the inner walls of the tube with a conducting coating and connecting this coating to the second anode, charges are prevented from accumulating on the walls of the tube.

20.5. The deflection sensitivity is the distance the spot moves for a given deflection voltage; it is generally given in centimeters per volt. The deflection factor is the reciprocal of the deflection sensitivity, and is generally given in volts per centimeter.

20.6. The deflection sensitivity is not so dependent on accurately maintained anode voltages when magnetic deflection is employed as it is when electrostatic deflection is employed.

20.7. Deflection sensitivity = 0.0225 cm/volt.

20.8. Deflection sensitivity = 0.03 cm/volt.

20.9. A saw-toothed wave form is generally used for a sweep voltage.

20.10. The return trace is the path of the electron spot across the screen from its end point back to its starting point.

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